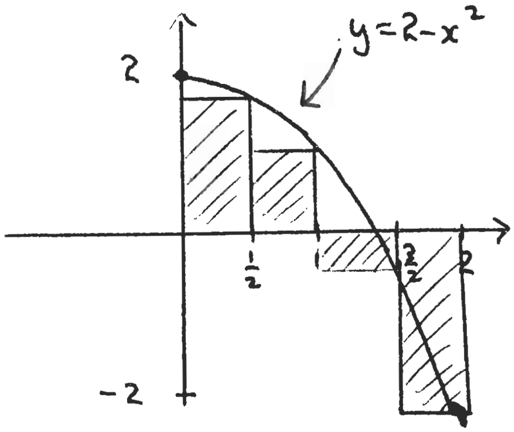
1. (a) Evaluate the Riemann sum for $f(x) = 2 - x^2$ taking $n = 4$, $a = 0$, $b = 2$ and the sample points to be right endpoints. Illustrate with a diagram what the Riemann sum represents. Marks [3]



$\Delta x = \frac{1}{2}$: sample pts (right endpts)

$x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$, $x_4 = 2$

Riemann sum

$$f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2}$$

$$= \frac{7}{4}\cdot\frac{1}{2} + 1\cdot\frac{1}{2} + \left(-\frac{1}{4}\right)\frac{1}{2} + (-2)\cdot\frac{1}{2}$$

$$= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

Represents: sum of areas of two rectangles above x-axis minus sum of areas of two rectangles below x-axis

(b) Evaluate $\int_0^2 (2 - x^2)dx$ from the limit definition of Riemann sums (using the right endpoints). Marks [4]

Divide $[0,2]$ into n equal subintervals so $\Delta x = \frac{2}{n}$ and right endpts are $x_1 = \frac{2}{n}$, $x_2 = \frac{4}{n}$, $\ldots$, $x_n = 2 = \frac{2n}{n}$. Thus $x_i = \frac{2i}{n}$

$$\therefore \int_0^2 (2 - x^2)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (2 - x_i^2)\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(2 - \frac{4i^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{2}{n} \left\{ 2n - \frac{\cancel{4}^{\,2}}{n^{\cancel{2}}}\cdot\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}_{\,3}} \right\}$$

$$= \lim_{n\to\infty} \left(4 - \frac{4}{3}\frac{(n+1)(2n+1)}{n^2} \right)$$

$$= \lim_{n\to\infty} \left(4 - \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \right) = 4 - \frac{8}{3} = \frac{4}{3}.$$

2 If $g(x) = \int_1^x \frac{dt}{t^3+1}$ find $g'(x)$ Marks [2]

$$g'(x) = \frac{1}{x^3+1}$$

3. Evaluate the following integrals:

(a) $\int_1^4 (5 - 2t + 3t^2)\, dt$ Marks [2]

$$= 5t - t^2 + t^3 \Big|_1^4 = (5(4) - 4^2 + 4^3) - (5 - 1 + 1)$$

$$= 68 - 5 = 63$$

(b) $\int_1^9 \frac{x-1}{\sqrt{x}}\, dx$ Marks [2]

$$= \int_1^9 (\sqrt{x} - \frac{1}{\sqrt{x}})\, dx = \int_1^9 (x^{1/2} - x^{-1/2})\, dx = \frac{2}{3}x^{3/2} - 2x^{1/2} \Big|_1^9$$

$$= (\frac{2}{3}(9)^{3/2} - 2(9)^{1/2}) - (\frac{2}{3} - 2) = 12 + \frac{4}{3} = \frac{40}{3}.$$

(c) $\int_0^{\frac{\pi}{4}} \sec\theta \tan\theta d\theta$ Marks [2]

$$= \sec\theta \Big|_0^{\pi/4}$$

$$= \sec\frac{\pi}{4} - \sec 0 = \sqrt{2} - 1$$

4. Evaluate the following integrals using the suggested substitution

(a) $\int x^4(5-x^5)^6\,dx$, $u = 5 - x^5$ Marks [2]

$du = -5x^4\,dx$, so

$\int = \int u^6\left(-\frac{du}{5}\right) = -\frac{1}{5}\int u^6\,du = -\frac{1}{5}\left(\frac{u^7}{7}\right) + C$

$= -\frac{1}{35}(5-x^5)^7 + C.$

(b) $\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$, $u = \sqrt{t}$ Marks [2]

$du = \frac{1}{2}\cdot\frac{1}{\sqrt{t}}\,dt$ so $\frac{dt}{\sqrt{t}} = 2\,du$

$\therefore \int = 2\int \cos u\,du = 2\sin u + C$

$= 2\sin(\sqrt{t}) + C$

(c) $\int_0^{\frac{\pi}{4}} \sec^3 x \tan x\,dx$, $u = \sec x$ Marks [3]

$du = \sec x \tan x\,dx$

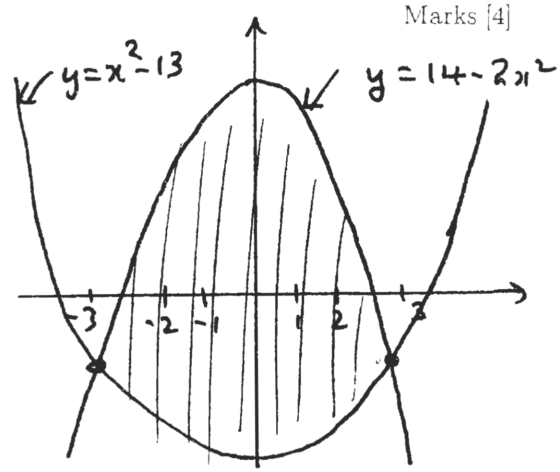
$x = 0 \Rightarrow u = 1$

$x = \frac{\pi}{4} \Rightarrow u = \sqrt{2}$

so $\int = \int_1^{\sqrt{2}} u^2\,du = \left.\frac{u^3}{3}\right|_1^{\sqrt{2}} = \frac{(\sqrt{2})^3 - 1}{3} = \frac{2\sqrt{2}-1}{3}.$

5. Sketch the region bounded by the curves $y = 14 - 2x^2$ and $y = x^2 - 13$ and find the area of this region.

Marks [4]



Intersection pts

$14 - 2x^2 = x^2 - 13$

$27 = 3x^2$

$9 = x^2 \therefore x = \pm 3$

so pts are $(\pm 3, -4)$.

Area $= \int_{-3}^{3} [(14 - 2x^2) - (x^2 - 13)]\,dx$

$= \int_{-3}^{3} (27 - 3x^2)\,dx = 27x - x^3 \Big|_{-3}^{3} = 54 - (-54) = 108$

or use $= 2\int_0^3 (27 - 3x^2)\,dx = 2(27x - x^3)\Big|_0^3 = 2(54) = 108$

6. A particle moves along a straight line with velocity $v(t) = 2t - 1$ for $0 \le t \le 2$. Find the net displacement and the distance travelled during the given time period.

Marks [4]

Net displacement $= \int_0^2 v(t)\,dt = \int_0^2 (2t-1)\,dt = t^2 - t\Big|_0^2 = 2$

Distance travelled $= \int_0^2 |v(t)|\,dt$.

Now $v(t) = 2t - 1 \geqslant 0 \iff t \geqslant \frac{1}{2}$

So Distance travelled $= \int_0^{\frac{1}{2}} -(2t-1)\,dt + \int_{1/2}^{2} (2t-1)\,dt$

$= t - t^2\Big|_0^{1/2} + t^2 - t\Big|_{1/2}^{2}$

$= \frac{1}{4} + (2 - (-\frac{1}{4})) = 2\frac{1}{2}$

7. Let $f(x) = \dfrac{2x+1}{3x-2}$. Show algebraically that f is one-to-one and find a formula for the inverse function f^{-1} Marks [4]

f is 1-1 : Suppose $f(x_1) = f(x_2)$. Then $\dfrac{2x_1+1}{3x_1-2} = \dfrac{2x_2+1}{3x_2-2}$

So $(2x_1+1)(3x_2-2) = (2x_2+1)(3x_1-2)$

$\therefore$ $6x_1x_2 + 3x_2 - 4x_1 - 2 = 6x_1x_2 + 3x_1 - 4x_2 - 2$

$\therefore$ $7x_2 = 7x_1$ so $x_2 = x_1$.

f^{-1}: Let $y = \dfrac{2x+1}{3x-2}$. So $y(3x-2) = 2x+1$

$\therefore$ $3xy - 2y = 2x + 1$

So $x(3y-2) = 2y+1$

Hence $x = \dfrac{2y+1}{3y-2}$ $\therefore$ $f^{-1}(y) = \dfrac{2y+1}{3y-2}$ or $f^{-1}(x) = \dfrac{2x+1}{3x-2}$.

8. Let $f(x) = 2x^3 + 3x + 1$. Show by Calculus that f has an inverse function and compute $(f^{-1})'(6)$. (Do not try and find an explicit formula for f^{-1}.) Marks [3]

$f'(x) = 6x^2 + 3 \geq 3 > 0$ so f is increasing on $\mathbb{R}$ and hence f is 1-1 on its domain

$(f^{-1})'(6) = \dfrac{1}{f'(a)}$ where $f(a) = 6$

ie $2a^3 + 3a + 1 = 6$

By inspection $a = 1$

so $(f^{-1})'(6) = \dfrac{1}{f'(1)}$. Now $f'(1) = 6 + 3 = 9$

so $(f^{-1})'(6) = \dfrac{1}{9}$.

9. Differentiate each of the following functions. (Simplify where possible.)

(a) $y = \dfrac{e^{2x}}{1+3x}$ Marks [2]

$$y' = \frac{(1+3x)(2e^{2x}) - e^{2x}(3)}{(1+3x)^2} = \frac{e^{2x}(2+6x-3)}{(1+3x)^2} = \frac{e^{2x}(6x-1)}{(1+3x)^2}$$

(b) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. Marks [2]

$$y' = \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2} = \frac{(e^{2x}+2+e^{-2x}) - (e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2}$$

$$\therefore y' = \frac{4}{(e^x+e^{-x})^2}.$$

10. Evaluate the following integrals:

(a) $\displaystyle\int e^x(4+e^x)^5\,dx$ Marks [2]

Let $u = 4+e^x$. Then $du = e^x dx$

So $\displaystyle\int = \int u^5 du = \frac{u^6}{6} + C = \frac{1}{6}(4+e^x)^6 + C$

(b) $\displaystyle\int_0^3 e^{-2x}\,dx$ Marks [2]

Let $u = -2x$, so $du = -2dx$. Also $x = 0 \Rightarrow u = 0$
$x = 3 \Rightarrow u = -6$

$$\therefore \int = \int_0^{-6} e^u \left(-\frac{du}{2}\right) = \frac{1}{2}\int_{-6}^{0} e^u du = \frac{1}{2} e^u \Big|_{-6}^{0}$$

$$= \frac{1}{2}(1 - e^{-6}) \approx .4975$$