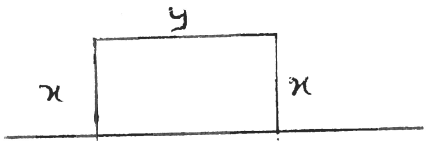
1. A farmer has 1000m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area? Marks [3]



Let x, y be as shown

We are given that

$2x + y = 1000$

We want to maximise Area, $A = xy$

so $A(x) = x(1000 - 2x) = 1000x - 2x^2$; $0 \le x \le 500$

critical points inside natural domain. $A'(x) = 0$

$1000 - 4x = 0$ so $x = 250$ is the only critical pt

$A(0) = 0$; $A(500) = 0$ so max occurs at $x = 250$

Dimensions of the field are 250×500.

2. Use Newton's method to find, correct to 4D, the root of $x^5 - x - 1 = 0$ in the interval $[1, 2]$. Start with $x_1 = 1$. Marks [4]

$f(x) = x^5 - x - 1$ so $f'(x) = 5x^4 - 1$

$x_1 = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{4} = 1.25$$

$$x_3 = 1.25 - \frac{f(1.25)}{f'(1.25)} = 1.25 - \frac{0.8018}{11.2070} = 1.1785$$

$$x_4 = 1.1785 - \frac{f(1.1785)}{f'(1.1785)} = 1.1785 - \frac{0.0947}{8.6447} = 1.1675$$

$$x_5 = 1.1675 - \frac{f(1.1675)}{f'(1.1675)} = 1.1675 - \frac{0.001624}{8.2896} = 1.1673$$

$\boxed{x_6 = 1.1673}$ is the root to 4D accuracy

3. The area under the curve $y = f(x)$, above the x-axis and bounded by the lines $x = a$ and $x = b$ is

$$A = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right).$$

Use this formula to find the area under $y = x^2 - x + 2$ from $x = 0$ to $x = 3$. Marks [4]

Here $a = 0$, $b = 3$ so $b - a = 3$; and

$$A = \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} f\left(\frac{3k}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left[\left(\frac{3k}{n}\right)^2 - \left(\frac{3k}{n}\right) + 2\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left[\frac{9}{n^2} k^2 - \frac{3}{n} k + 2\right]$$

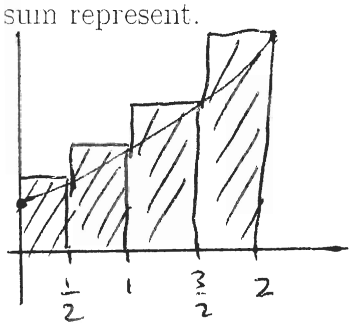
$$= \lim_{n\to\infty} \left\{ \frac{27}{n^3} \sum_{k=1}^{n} k^2 - \frac{9}{n^2} \sum_{k=1}^{n} k + \frac{6}{n} n \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} - \frac{9}{n^2} \frac{n(n+1)}{2} + 6 \right\}$$

$$= \lim_{n\to\infty} \frac{27}{\cancel{n^3}} \frac{\cancel{n^3}(1)(1+\frac{1}{n})(2+\frac{1}{n})}{6} - \frac{9(n+1)}{n \cdot 2} + 6 \}$$

$$= 9 - \frac{9}{2} + 6 = \frac{21}{2}$$

4. (a) Find the approximation to $\int_0^2 (1+x^2)\,dx$ using a Riemann sum with right endpoints and $n = 4$. Illustrate with a diagram what the Riemann sum represent. Marks [3]



$n = 4 \quad \Delta x = \frac{b-a}{4} = \frac{1}{2}$

So right endpts are $\frac{1}{2}, 1, \frac{3}{2}, 2$

Riemann sum represents the total area of the four rectangles shown

$$\text{Approximation} = \frac{1}{2}\left\{1+\left(\tfrac{1}{2}\right)^2 + 1+(1)^2 + 1+\left(\tfrac{3}{2}\right)^2 + 1+2^2\right\}$$

$$= \frac{1}{2}\left\{4 + \frac{1}{4} + 1 + \frac{9}{4} + 4\right\}$$

$$= \frac{23}{4}$$

(b) Evaluate the exact answer to $\int_0^2 (1+x^2)\,dx$. Marks [1]

$$\int_0^2 (1+x^2)\,dx = x + \frac{x^3}{3}\Big|_0^2 = \left(2+\frac{8}{3}\right) - 0 = \frac{14}{3}$$

5. A particle moves in a straight line and has acceleration given by $a(t) = 2t+3$. Its initial velocity is $v(0) = -3$ cm/sec and its initial displacement is $s(0) = 8$ cm. Find its position function $s(t)$. Marks [2]

$$v(t) = \int a(t)\,dt = \int (2t+3)\,dt = t^2 + 3t + C$$

$t = 0 \Rightarrow -3 = 0 + 0 + C \therefore C = -3$

so $v(t) = t^2 + 3t - 3$

$$s(t) = \int v(t)\,dt = \int (t^2+3t-3)\,dt = \frac{t^3}{3} + \frac{3t^2}{2} - 3t + D$$

$t = 0 \Rightarrow s(0) = 8 \therefore D = 8$ so $s(t) = \frac{t^3}{3} + \frac{3t^2}{2} - 3t + 8$

6. Evaluate the following integrals:

(a) $\int_1^2 \left(\frac{3}{t^4}\right)\, dt$ Marks [2]

$$= 3\int_1^2 t^{-4}\,dt = 3 \left.\frac{t^{-3}}{-3}\right|_1^2 = \left.-t^{-3}\right|_1^2 = -\frac{1}{8} + 1 = \frac{7}{8}$$

(b) $\int_0^3 (x+1)(2x+1)\, dx$ Marks [2]

$$= \int_0^3 (2x^2 + 3x + 1)\,dx = \left.\frac{2}{3}x^3 + 3\frac{x^2}{2} + x\right|_0^3$$

$$= \frac{2}{3}(3)^3 + 3\left(\frac{3^2}{2}\right) + 3 - 0 = \frac{69}{2} = 34\tfrac{1}{2}$$

(c) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (3\cos x + 2\sin x)\, dx$ Marks [2]

$$= \left.3\sin x - 2\cos x\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(3\sin\frac{\pi}{2} - 2\cos\frac{\pi}{2}\right) - \left(3\sin\frac{\pi}{4} - 2\cos\frac{\pi}{4}\right)$$

$$= (3 - 0) - \left(\frac{3}{\sqrt{2}} - \frac{2}{\sqrt{2}}\right) = 3 - \frac{1}{\sqrt{2}}.$$

7. Evaluate the following integrals using the suggested substitution

(a) $\int (x^2+3)^7 x\, dx$, $u = x^2+3$ Marks [2]

$du = 2x\,dx$ so $x\,dx = \frac{du}{2}$

Hence $\int$ becomes $\int u^7 \frac{du}{2} = \frac{1}{2}\int u^7 du = \frac{1}{2}\frac{u^8}{8} + C$

$= \frac{(x^2+3)^8}{16} + C$

(b) $\int_1^2 \frac{dx}{(3-5x)^2}$, $u = 3-5x$ Marks [2]

$du = -5\,dx$ so $dx = -\frac{du}{5}$

Also $x = 1 \Rightarrow u = -2$; $x = 2 \Rightarrow u = -7$

Hence $\int$ becomes $\int_{-2}^{-7} u^{-2}\left(-\frac{du}{5}\right) = \frac{1}{5}\int_{-7}^{-2} u^{-2}\,du$

$= \frac{1}{5}(-u^{-1})\Big|_{-7}^{-2} = -\frac{1}{5}\left\{(-2)^{-1} - (-7)^{-1}\right\} = \frac{1}{5}\left(\frac{1}{2} - \frac{1}{7}\right) = \frac{1}{14}$

(c) $\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}$, $u = 1+\tan t$ Marks [2]

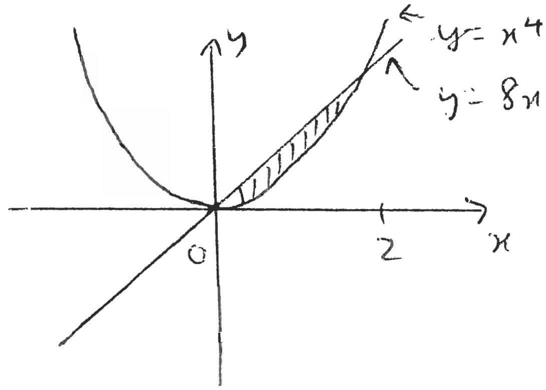
$du = \sec^2 t\,dt$ so $\frac{dt}{\cos^2 t} = du$

Hence $\int$ becomes $\int \frac{du}{u^{1/2}} = \int u^{-1/2}\,du$

$= \frac{u^{1/2}}{1/2} + C$

$= 2\sqrt{u} + C = 2\sqrt{1+\tan t} + C$

8. Sketch the region bounded by the curves $y = x^4$ and $y = 8x$ and find the area of this region. Marks [3]



$x^4 = 8x$

$x(x^3 - 8) = 0$

$x = 0$ or $x = 2$

So int. pts are $(0,0)$ and $(2,16)$

Area enclosed $= \displaystyle\int_0^2 (8x - x^4)\,dx = 4x^2 - \frac{x^5}{5}\Big|_0^1 = 4 - \frac{1}{5} = \frac{19}{5}$

9. Let $f(x) = \dfrac{1-x}{2+x}$.

(a) Show **algebraically** that f is one-to-one and find a formula for the inverse function f^{-1}. Marks [3]

Suppose $f(x_1) = f(x_2)$ So $\dfrac{1-x_1}{2+x_1} = \dfrac{1-x_2}{2+x_2}$

Hence $(1-x_1)(2+x_2) = (1-x_2)(2+x_1)$

So $2 - 2x_1 + x_2 - x_1x_2 = 2 - 2x_2 + x_1 - x_1x_2$

$\therefore$ $3x_2 = 3x_1$ so $x_2 = x_1$, as required

$y = \dfrac{1-x}{2+x} \Rightarrow (2+x)y = 1-x \Rightarrow x(y+1) = 1-2y$

Thus $x = \dfrac{1-2y}{1+y} = f^{-1}(y)$.

(b) Find the domain and range of f. Marks [1]

Domain of $f = \{x : x \neq -2\} = (-\infty, -2) \cup (-2, \infty)$

Range of $f = \{y : y \neq -1\} = (-\infty, -1) \cup (-1, \infty)$

10. Evaluate

(a) $\frac{d}{dx}\left(x^4 e^{-3x}\right)$ Marks [2]

$= 4x^3 e^{-3x} + x^4 (e^{-3x})(-3)$

$= e^{-3x} (4x^3 - 3x^4)$

(b) $\int_0^1 (e^x + e^{-x})^2 \, dx.$ Marks [2]

$= \int_0^1 (e^{2x} + 2e^x \cdot e^{-x} + e^{-2x}) \, dx$

$= \int_0^1 (e^{2x} + 2 + e^{-2x}) \, dx$

$= \frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2} \Big|_0^1 = \left(\frac{e^2}{2} + 2 - \frac{e^{-2}}{2}\right) - \left(\frac{1}{2} - \frac{1}{2}\right)$

$= \frac{e^2}{2} + 2 - \frac{e^{-2}}{2}$