TRIGONOMETRY

1 Acute Angles

In secondary school the trigonometric functions are first introduced for acute angles (that is, angles between 0° and 90°) as follows:

Suppose θ is one of the acute angles of the right-angled triangle ΔABC as shown in Figure 1.

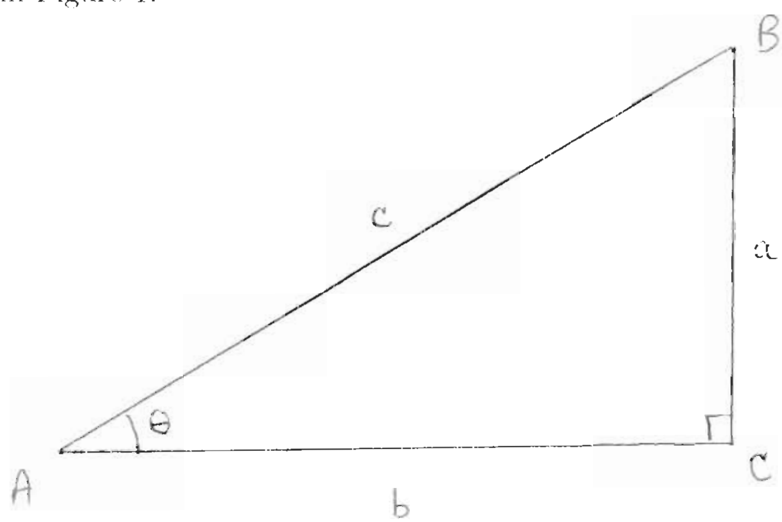


Figure 1

We put $a = BC$, $b = AC$ and $c = AB$. Note that $c^2 = a^2 + b^2$ (Pythagoras). Then we define

$$\begin{cases} \sin\theta = \dfrac{a}{c} \quad , \quad \cos\theta = \dfrac{b}{c} \quad , \quad \tan\theta = \dfrac{a}{b} \\ \\ \csc\theta = \dfrac{c}{a} \quad , \quad \sec\theta = \dfrac{c}{b} \quad , \quad \cot\theta = \dfrac{b}{a} \end{cases} \tag{1.1}$$

Note that these six functions do not depend on the size of the given triangle, since similar triangles will have the same ratios of corresponding sides. Note also that there are various connections between the six functions. For example we have that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$ and $\csc\theta = \dfrac{1}{\sin\theta}$. Finally we can see that $90° - \theta$ corresponds to $\angle ABC$, so that $\sin(90° - \theta) = \cos\theta$ and $\cos(90° - \theta) = \sin\theta$.

There are a few special acute angles for which we can determine exactly the values of each of the six trig. functions.

Example 1 *Let $\theta = 45°$. Then the triangle to consider is the isoceles right-angled triangle shown in Figure 2.*

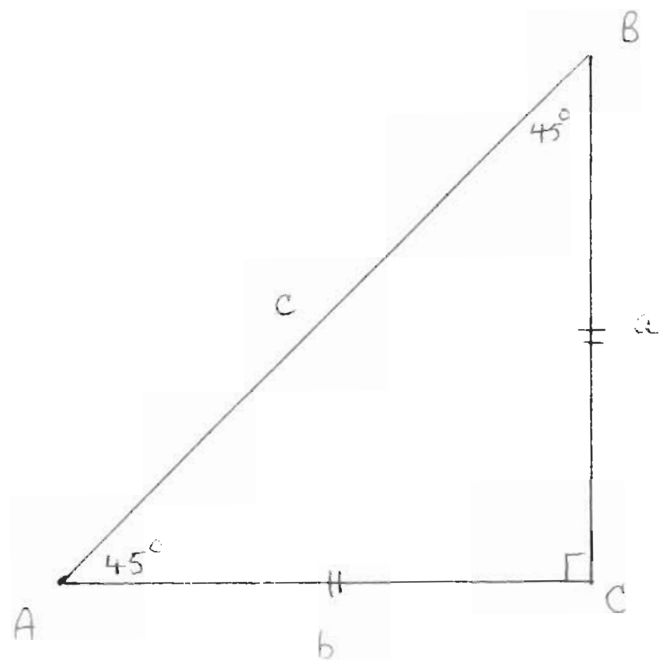


Figure 2

Thus $b = a$ and $c^2 = a^2 + b^2 = a^2 + a^2 = 2a^2$. So $c = \sqrt{2}a$. Thus

$$\sin 45° = \frac{a}{c} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}, \cos 45° = \frac{b}{c} = \frac{a}{c} = \frac{1}{\sqrt{2}},$$

$$\tan 45° = \frac{a}{b} = 1, \cot 45° = \frac{b}{a} = 1,$$

$$\sec 45° = \frac{1}{\cos 45°} = \sqrt{2}, \csc 45° = \frac{1}{\sin 45°} = \sqrt{2}.$$

Example 2 *Let $\theta = 30^\circ$. We consider ΔABC in Figure 3.*

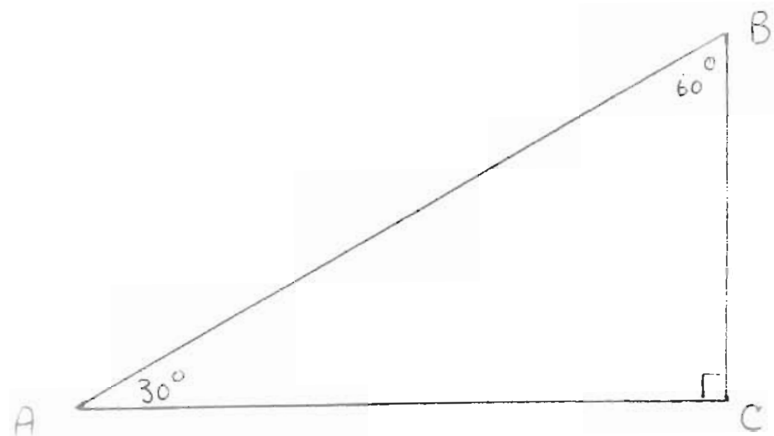


Figure 3

Note that $\angle ABC = 60^\circ$, which corresponds to one angle of an equilateral triangle, so let's look at Figure 4.

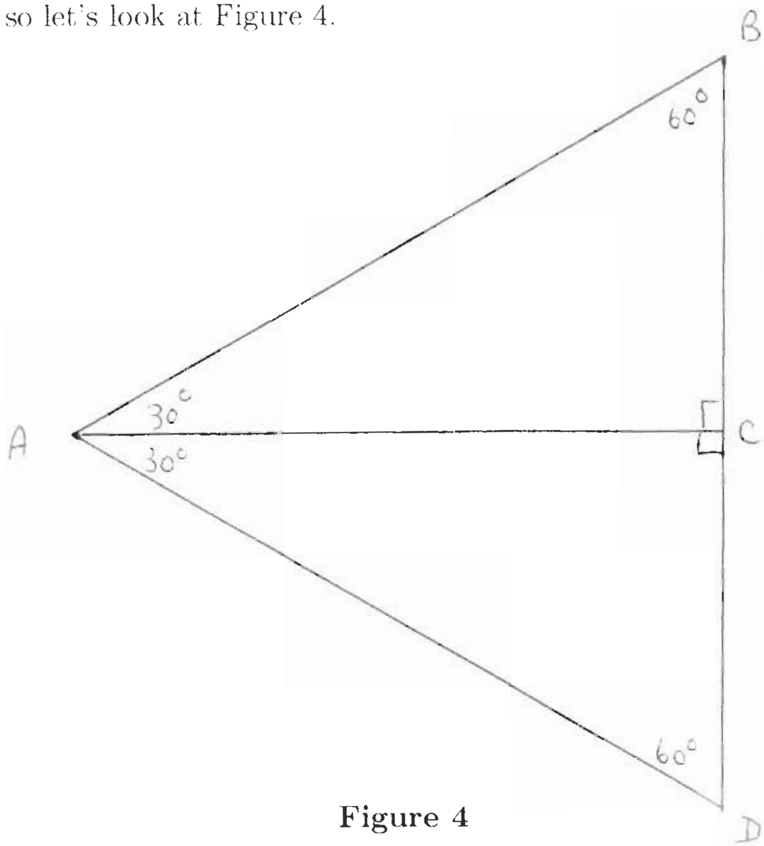


Figure 4

ΔABD is equilateral and $BC = CD$. If we take $AB = 2$ then $BC = 1$ and so $AC = \sqrt{3}$ by Pythagoras. Then $\sin 30° = \dfrac{BC}{AB} = \dfrac{1}{2}$, $\cos 30° = \dfrac{AC}{AB} = \dfrac{\sqrt{3}}{2}$. Thus $\tan 30° = \dfrac{1}{\sqrt{3}}$, $\cot 30° = \sqrt{3}$, $\csc 30° = 2$, $\sec 30° = \dfrac{2}{\sqrt{3}}$.

Note also we can compute the values of the trig. functions for $\theta = 60°$ from the same diagram! Thus

$$\sin 60° = \frac{AC}{AB} = \frac{\sqrt{3}}{2}, \cos 60° = \frac{BC}{AB} = \frac{1}{2};$$

$$\tan 60° = \sqrt{3}, \cot 60° = \frac{1}{\sqrt{3}}, \csc 60° = \frac{2}{\sqrt{3}}, \sec 60° = 2. \qquad \blacksquare$$

For other acute angles we usually need to use a calculator for the computation of the values of the various trig. functions, and then we are able to solve problems like the following.

Example 3 *Find the value of y to four decimals in the triangle of Figure 5.*

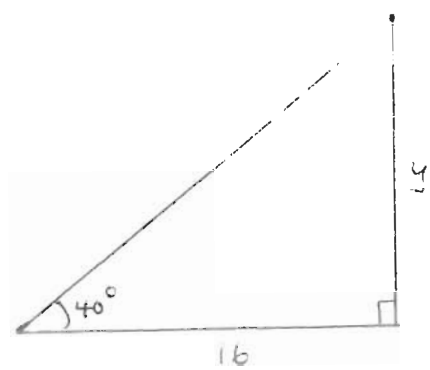


Figure 5

Solution:

$$\begin{aligned} \tan 40° &= \frac{y}{16}, \text{ so } y = 16 \tan 40° \\ &= 16(0.83910) \\ &= 13.4256 \end{aligned} \qquad \blacksquare$$

2 General Angles

If we place our right-angled triangle ΔOPQ in the first quadrant of the (x, y) plane as shown in Figure 6, so that one vertex is at the origin, and another is on the x-axis, then we have P at (x, y) so Q is at $(x, 0)$. Also $PQ = y$, $OQ = x$. Finally we set $OP = r$.

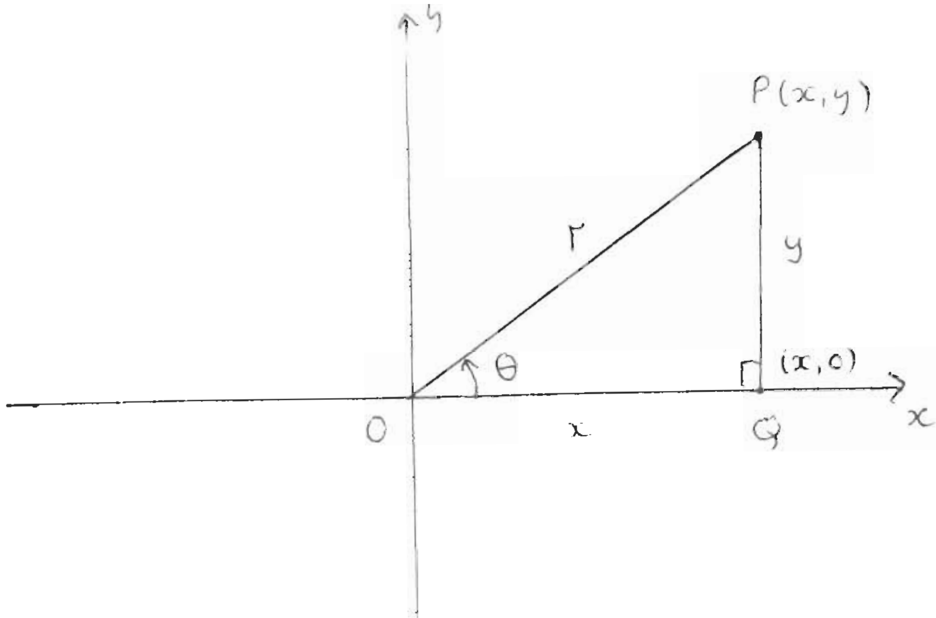


Figure 6

$$\text{Thus} \begin{cases} \sin\theta = \dfrac{y}{r} \quad , \quad \cos\theta = \dfrac{x}{r} \quad , \quad \tan\theta = \dfrac{y}{x} \quad , \\ \\ \csc\theta = \dfrac{r}{y} \quad , \quad \sec\theta = \dfrac{r}{x} \quad , \quad \cot\theta = \dfrac{x}{y} \end{cases} \tag{2.1}$$

Note that this way of looking at the trig functions allows us to define them for any angle at all. Suppose C is a circle of radius r with centre at O. If θ is an arbitrary positive angle then we imagine wrapping around the circle in a counterclockwise direction, starting from the point on C on the positive x-axis and terminating at the point P on the circle C. Figure 7 illustrates a few scenarios.

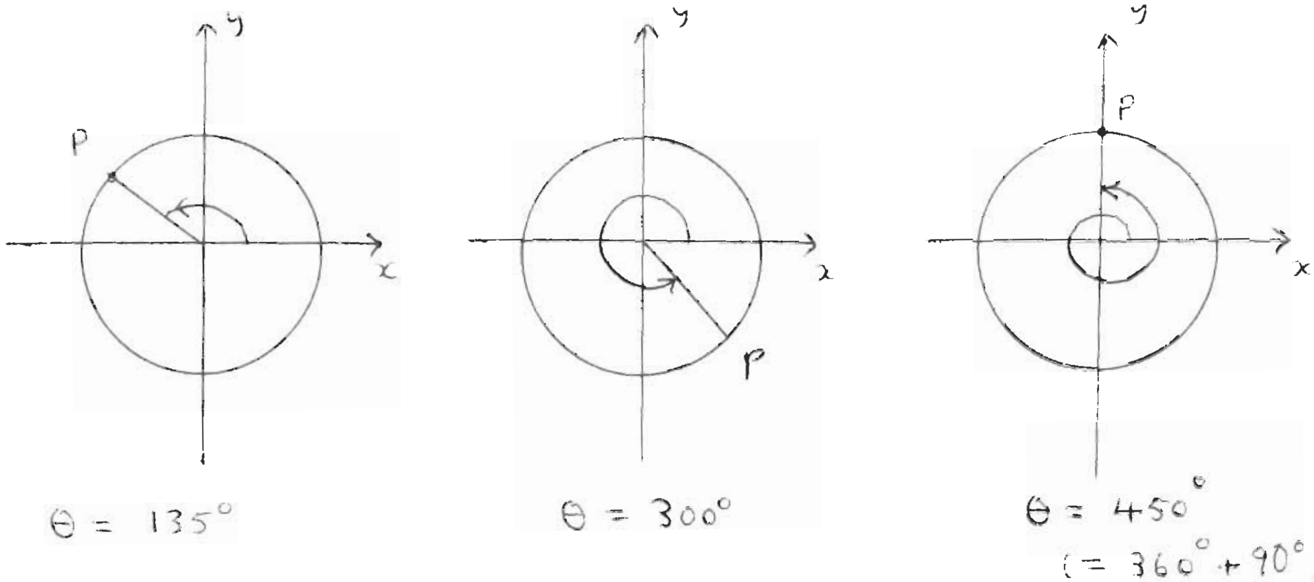


Figure 7

If θ is a negative angle we just wrap around the circle in the clockwise direction instead. Figure 8 illustrates a couple of situations of this type.

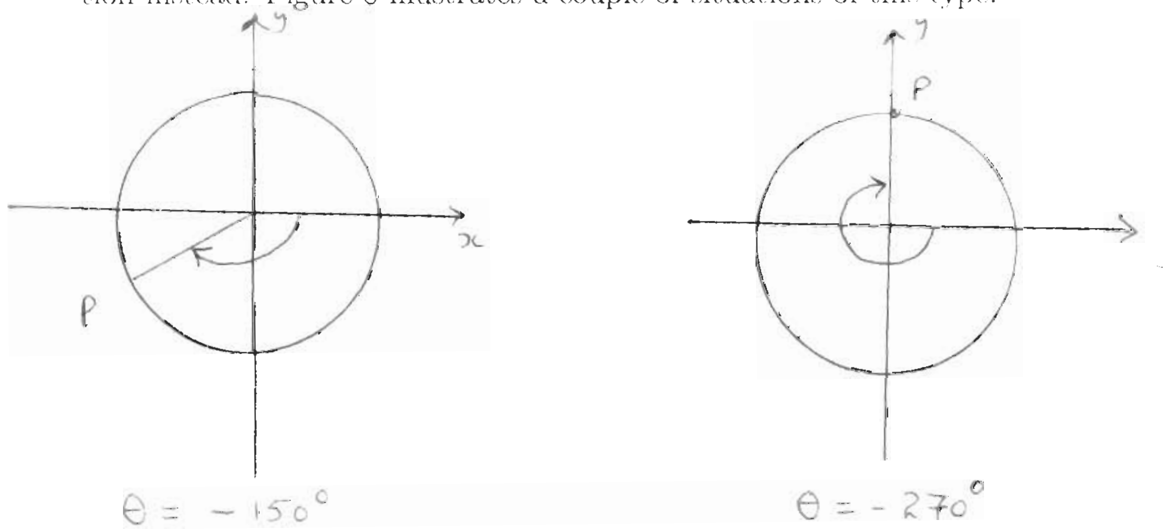


Figure 8

Now we can use equation 2.1 to figure out the values of the various trig. functions at these general angles.

Example 4 *Let $\theta = 90°$. Thus we have $P = (0, r)$, as in Figure 9.*

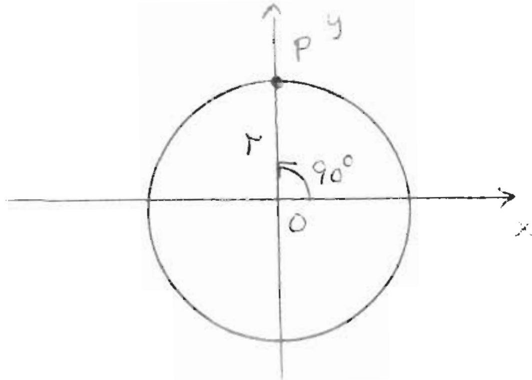


Figure 9

So $x = 0$, $y = r$ and we obtain sin $90° = \dfrac{y}{r} = 1$, $\cos 90° = \dfrac{x}{r} = 0$, $\tan 90°$ *and* $\sec 90°$ *are undefined,* $\cot 90° = 0$, $\csc 90° = 1$. ∎

Example 5 *Let $\theta = 135°$. We refer to Figure 10 and also refer back to Example 1 and Figure 2.*

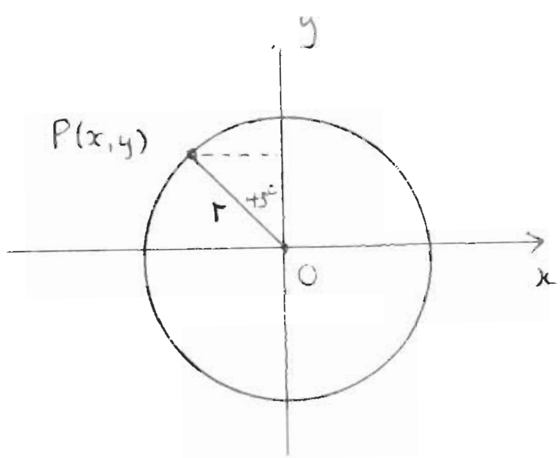


Figure 10

From the diagram we see that $x^2 + y^2 = r^2$ and $x = -y$. So $y = \frac{r}{\sqrt{2}}$ and $x = -\frac{r}{\sqrt{2}}$. Then $\sin 135° = \frac{y}{r} = \frac{1}{\sqrt{2}}$, $\cos 135° = \frac{x}{r} = -\frac{1}{\sqrt{2}}$. We can then easily obtain the calues of the other four trig functions.

We can then complete the following table of values

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1

θ	$210°$	$225°$	$240°$	$270°$	$300°$	$315°$	$330°$	$360°$
$\sin\theta$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0
$\cos\theta$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

Example 6 *If* $\tan\theta = \frac{2}{3}$ *and* $180° < \theta < 270°$, *find* $\cos\theta$.

Solution: Referring to Figure 11, since $\tan\theta = \frac{2}{3} = \frac{-2}{-3}$,

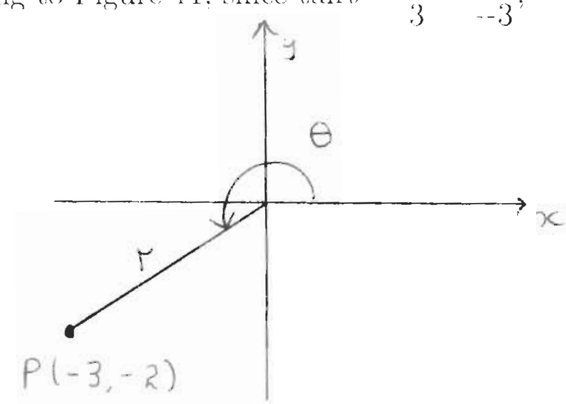


Figure 11

we can take P as $(-3, -2)$, so $r^2 = 13$. Thus $r = \sqrt{13}$ and $\cos\theta = \frac{x}{r} = -\frac{3}{\sqrt{13}}$. ∎

Note also from 2.1 that:
$\sin\theta > 0$ in quadrants I, II;
$\cos\theta > 0$ in quadrants I, IV;
$\tan\theta > 0$ in quadrants I, III.

We can summarise this information in Figure 12:

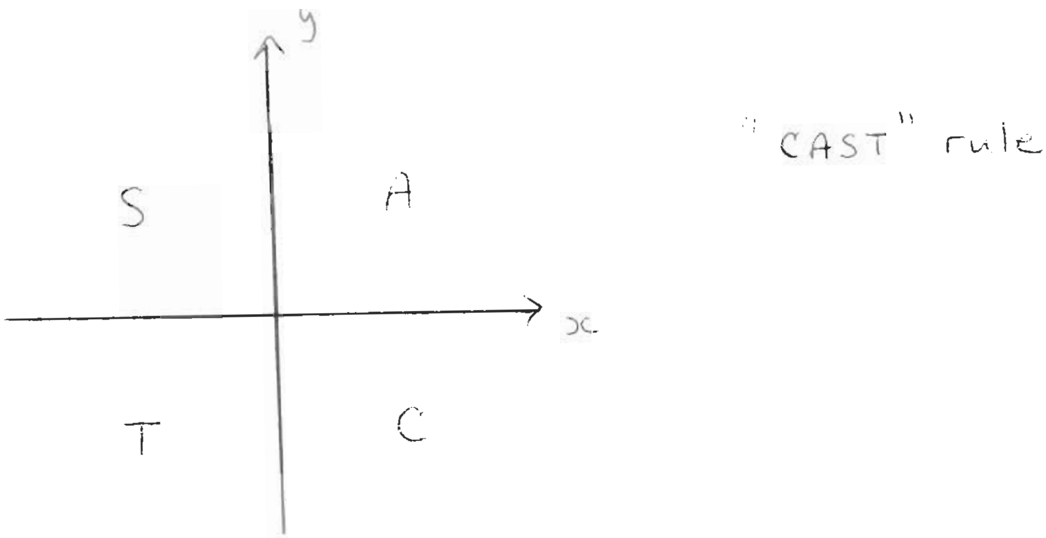


Figure 12

The following identities are all derived from the definition 2.1 (see Appendix D page A28):

$$\begin{cases} \sin^2\theta + \cos^2\theta = 1 \\ \sec^2\theta = 1 + \tan^2\theta \\ \csc^2\theta = 1 + \cot^2\theta \end{cases} \tag{2.2}$$

We can use these identities to help simplify expressions.

Example 7 *Also from the definition we see that*

$$\begin{cases} \sin(\theta + 360°) = \sin\theta; \\ \cos(\theta + 360°) = \cos\theta; \\ \tan(\theta + 180°) = \tan\theta. \\ \sin(-\theta) = -\sin\theta \\ \cos(-\theta) = \cos\theta \end{cases} \tag{2.3}$$

We can use the trig. functions to compute the length of the side of a general triangle.

Law of Cosines:

If ΔABC is a triangle with sides a, b, c and $\angle ACB = \theta$ as in Figure 13, then $c^2 = a^2 + b^2 - 2ab\cos\theta$

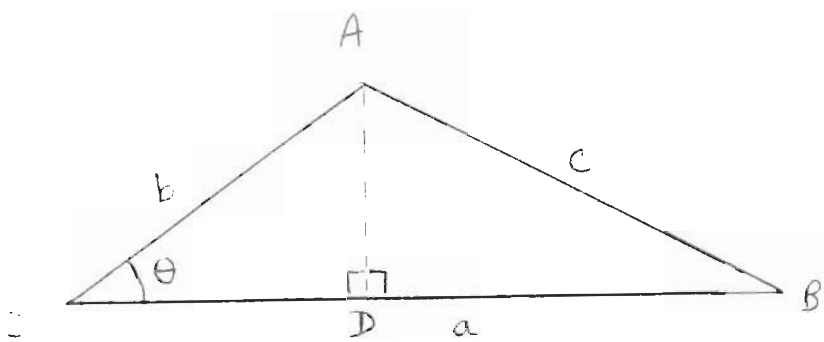


Figure 13

Proof: If we drop the perpendicular from A so that ΔACD and ΔADB are both right-angled triangles, then

$$\begin{aligned} AD &= b\sin\theta \text{ and } BD = a - b\cos\theta. \\ \text{Thus } c^2 &= AD^2 + BD^2 = (b\sin\theta)^2 + (a - b\cos\theta)^2 \\ &= b^2\sin^2\theta + a^2 - 2ab\cos\theta + b^2\cos^2\theta \\ &= b^2(\sin^2\theta + \cos^2\theta) + a^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta, \text{ as required.} \end{aligned}$$ ∎

Example 8 *If ΔABC has $a = 3$, $b = 5$, $\angle ACB = \theta = 42°$, find c.*

Solution: From the Law of Cosines we obtain

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\theta \\ &= 3^2 + 5^2 - 2\cdot 3\cdot 5\cos 42° \\ &= 34 - 30(-.7431) = 11.707 \\ \text{Thus } c &= 3.4216 \end{aligned}$$ ∎

There are addition formulas:

$$\begin{cases} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B \end{cases} \tag{2.4}$$

By using the facts that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$, we also have

$$\begin{cases} \sin(A-B) &= \sin A \cos B - \cos A \sin B \\ \cos(A-B) &= \cos A \cos B + \sin A \sin B \end{cases} \tag{2.5}$$

Example 9 *Use the addition formulas to evaluate* $\sin(75°)$ *and* $\cos(75°)$ *without using a calculator.*

Solution: Let $A = 45°$ and $B = 30°$. From 2.4 we obtain

$$\begin{aligned} \sin 75° &= \sin(45° + 30°) \\ &= \sin 45° \cos 30° + \cos 45° \sin 30° \\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \quad \text{, using Examples 1, 2} \\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \text{Similarly } \cos 75° &= \cos(45° + 30°) \\ &= \cos 45° \cos 30° - \sin 45° \cdot \sin 30° \\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \end{aligned}$$

∎

If we put $A = B$ in the addition formulas 2.4 we get

$$\sin 2A = 2 \sin A \cos A \text{ and } \cos 2A = \cos^2 A - \sin^2 A \tag{2.6}$$

We can use $\cos^2 A + \sin^2 A = 1$ to then derive

$$\begin{aligned} \cos 2A &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned} \tag{2.7}$$

$$\text{and also } \sin^2 A = \frac{1 - \cos 2A}{2}, \cos^2 A = \frac{1 + \cos 2A}{2} \tag{2.8}$$

Example 10 *Simplify* $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta}$.

Solution:

$$\begin{aligned}
\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} &= \frac{\sin\theta + \sin^2\theta + \cos^2\theta}{\cos\theta(1+\sin\theta)} \\
&= \frac{1+\sin\theta}{\cos\theta(1+\sin\theta)} \\
&= \frac{1}{\cos\theta} \\
&= \sec\theta
\end{aligned}$$

3 Radian measures of angles

In order to use the trigonometric functions in conjuction with the standard functions (polynomials, rational functions, exponentials, logarithms) we have to measure angles differently – to allow real numbers as angles.

Recall that the circumference of a circle of radius 1 has length 2π. So we make the real number 2π correspond to 360° using the word radian to indicate the correspondence.

$$\begin{aligned}
2\pi \text{ radians} &= 360^\circ \text{ or} \\
\pi \text{ radians} &= 180^\circ \\
1 \text{ radian} &= \frac{180^\circ}{\pi} \approx 57.3^\circ \\
1^\circ &= \frac{\pi}{180} \text{ radians} \approx 0.017 \text{ radians}
\end{aligned}$$

Example 11 *1. Find the radian measure of* 75°.

2. Express $\frac{5\pi}{4}$ *radians in degrees.*

Solution:

1.
$$\begin{aligned} \text{Since } 180^\circ &= \pi \text{ radians we obtain} \\ 75^\circ &= \frac{75^\circ}{180^\circ}(180^\circ) \\ &= \frac{5}{12}(\pi \text{ radians}) \\ &= \frac{5\pi}{12} \text{ radians} \end{aligned}$$

2. $\frac{5\pi}{4}$ radians $= \frac{5}{4}(180^\circ) = 5(45^\circ) = 225^\circ$. ∎

If θ is a number then $\sin\theta$ means the sine of the angle whose radian measure is θ. Thus for example $\sin 2\pi = 0$. Likewise $\cos 2\pi = 1$ and $\cos\pi = -1$. We can then redo the table of values from page 8 to obtain

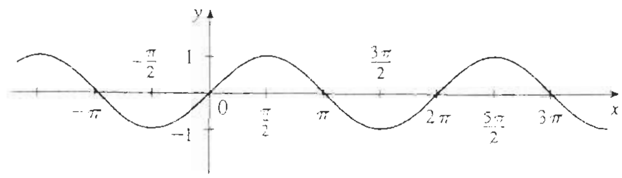
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1

and a similar second table. We can also do the same calculations for the other five trigonometric functions and summarize the information on graphs in Figure 14 (see next page).

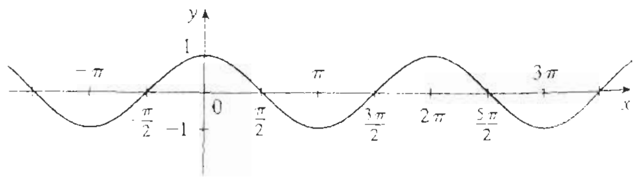
Example 12 *Find all values of x in $[0, 2\pi]$ such that* $\sin x = \sin 2x$.

Solution: Using (2.6) we can rewrite the given equation as $\sin x = 2\sin x\cos x$. Thus $\sin x(1 - 2\cos x = 0$.

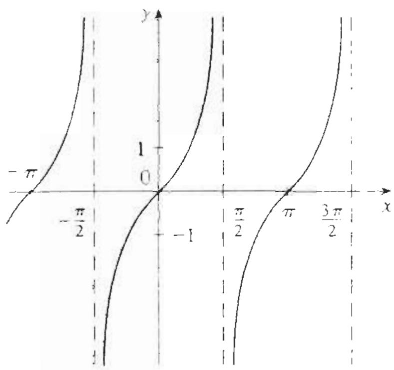
$$\begin{aligned} \text{Now } \sin x = 0 &\implies x = 0, \pi, 2\pi, \text{ and} \\ \cos x = \frac{1}{2} &\implies x = \frac{\pi}{3}, \frac{5\pi}{3}. \end{aligned}$$ ∎



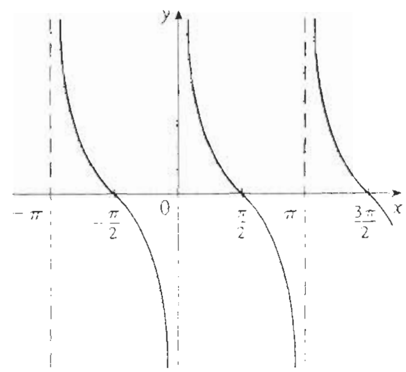
$y = \sin x$



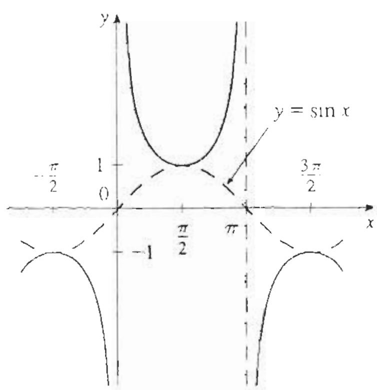
$y = \cos x$



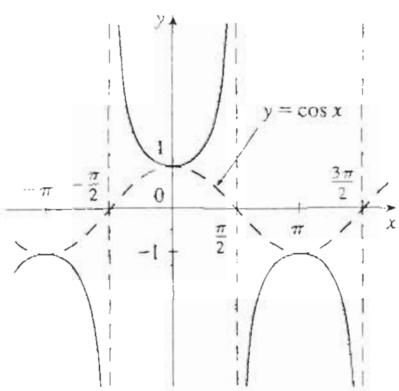
$y = \tan x$



$y = \cot x$



$y = \csc x$



$y = \sec x$

Figure 14