SOLUTIONS

MATH/COSC 3416 EA – TERM TEST 2

Thursday, March 24, 2011, 8:30 am

Time Allowed: 75 minutes *Total Marks:* 35 *Instructor:* Barry G. Adams

Name: ____________________

Question 1, Trapezoidal Rule

The composite trapezoidal rule $T(f,h)$ with error formula is given by

$$\int_a^b f(x)\,dx = T(f,h) - \frac{b-a}{12}h^2 f''(\xi), \quad \text{where } a \leqslant \xi \leqslant b \text{ and}$$

$$T(f,h) = \frac{h}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right]$$

Here $x_0 = a$, $x_n = b$, $x_k = a + kh$, and $h = (b-a)/n$.

(a) (5 marks) Evaluate the integral $\int_0^{1/10} \frac{dx}{x+1}$ using the composite trapezoidal rule with 5 steps (subintervals).

> **Solution:** Using $h = 0.1/5 = 1/50$
>
> $$\begin{aligned} T(f,h) &= \frac{1}{100}\left[1 + 2\left\{\frac{1}{1+1/50} + \frac{1}{1+2/50} + \frac{1}{1+3/50} + \frac{1}{1+4/50}\right\} + \frac{1}{1+5/50}\right] \\ &= 0.095316 \quad \text{(exact answer is 0.095310)} \end{aligned}$$

(b) (2 marks) Use the error formula to estimate the error in your answer to Question 1(a).

> **Solution:** Calculate the second derivative: $f(x) = (1+x)^{-1}$, $f'(x) = -(1+x)^{-2}$, $f''(x) = 2(1+x)^{-3}$. The maximum value of $|f''(x)|$ on the interval $[0, 1/10]$ is 2 so the absolute error is
>
> $$|E| = \frac{0.1}{12}\left(\frac{1}{50}\right)^2 |f''(\xi)| \leqslant \frac{1}{120}\left(\frac{1}{50}\right)^2 2 = 0.67 \times 10^{-5}$$

(c) (2 marks) For the integral in Question 1(a), how many steps (subintervals) would be needed to obtain an absolute error less than or equal to 0.5×10^{-6}.

Solution: for step size h the absolute error must satisfy

$$|E| \leqslant \left(\frac{0.1}{12}\right) h^2 2 = \frac{1}{60} h^2 \leqslant 0.5 \times 10^{-6}$$

Therefore $h^2 \leqslant 30 \times 10^{-6}$ so $h \leqslant 0.005477$. Therefore $n = 0.1/h = 0.1/0.005477 = 18.26$ and a suitable value of n is 19.

Question 2, Simpson's Rule

The composite Simpson's rule $S(f,h)$ with error formula is given by

$$\int_a^b f(x)\,dx = S(f,h) - \frac{b-a}{180} h^4 f^{(4)}(\xi), \quad \text{where } a \leqslant \xi \leqslant b \text{ and}$$

$$S(f,h) = \frac{h}{3}\left[f(a) + 2\sum_{k=1}^{n/2-1} f(x_{2k}) + 4\sum_{k=1}^{n/2} f(x_{2k-1}) + f(b)\right]$$

Here n is even, $x_0 = a$, $x_n = b$, $x_k = a + kh$, and $h = (b-a)/n$.

(a) (5 marks) Use Simpson's rule with $n = 4$, $h = 1/4$ to estimate the integral $\int_0^1 e^{-x}\,dx$

Solution: Using $h = 0.25$

$$\begin{aligned}\int_0^1 e^{-x}\,dx &= \frac{1}{12}\left[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)\right] \\ &= \frac{1}{12}\left[1 + 4e^{-0.25} + 2e^{-0.5} + 4e^{-0.75} + e^{-1}\right] \\ &= 0.63213 \quad \text{(exact answer is 0.63212)}\end{aligned}$$

(b) (2 marks) For the integral in Question 2(a), how many steps (subintervals) would be needed to obtain an absolute error less than or equal to 0.5×10^{-6}.

Solution: The maximum of $f^{(4)}(x)$ on the interval $[0,1]$ is 1 so the absolute error satisfies

$$|E| \leqslant \frac{h^4}{180} \leqslant 0.5 \times 10^{-6}$$

Therefore $h^4 \leqslant 9 \times 10^{-5}$ and $h \leqslant 0.09740$. The number of steps is $n = 1/h = 10.267$ so a suitable value for n is 11.

(c) (2 marks) Derive the composite Simpson formula $S(f,h)$ from the composite trapezoidal formula $T(f,h)$ using Richardson extrapolation.

Solution:

$$\int_a^b f(x)\,dx = T(f,h) + K_1h^2 + K_2h^4 + \cdots$$
$$\int_a^b f(x)\,dx = T(f,2h) + 4K_1h^2 + 16K_2h^4 + \cdots$$

Multiply the first equation by 4 and subtract from the second equation to eliminate the h^2 term and get

$$3\int_a^b f(x)\,dx = 4T(f,h) - T(f,2h) + O(h^4)$$

Define

$$S(f,h) = \frac{4T(f,h) - T(f,2h)}{3}$$

and use the composite trapezoidal rule to get

$$\begin{aligned}
T(f,h) &= \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{2n-1} + y_{2n}] \\
T(f,2h) &= \frac{2h}{2}[y_0 + 2y_2 + 2y_4 + 2y_6 + \cdots + 2y_{2n-2} + y_{2n}] \\
S(f,h) &= \frac{2h}{3}[y_0 + 2y_1 + \cdots + 2y_{2n-1} + y_{2n}] - \frac{h}{3}[y_0 + 2y_2 + 2y_4 + \cdots + 2y_{2n-2} + y_{2n}] \\
&= \frac{h}{3}[y_0 + 2(y_2 + y_4 + \cdots + y_{2n-2}) + 4(y_1 + y_3 + \cdots + y_{2n-1}) + y_{2n}]
\end{aligned}$$

Question 3, Simpson's Rule Algorithm

(6 marks) Write a Matlab function for Simpson's rule with first line

```
function [sum] = simp(f, a, b, n)
```

where `f` is the function handle, `a` and `b` are the lower and upper integration limits, `n` is the number of steps (even nunber), and the approximation to the integral is returned as `sum`.

Solution:

```
function [sum] = simp(f, a, b, n)
if rem(n,2) == 1
   error('n must be even');
end

h = (b - a) / n;

sumEven = 0.0;
for k = 1 : n/2 - 1
   sumEven = sumEven + f(a + 2*k*h);
end

sumOdd = 0.0;
for k = 1 : n/2
   sumOdd = sumOdd + f(a + (2*k-1)*h);
end

sum = h*(f(a) + f(b) + 2.0*sumEven + 4.0*sumOdd) / 3.0;

end
```

Question 4, Romberg Method

(a) (4 marks) Given the values in column 0 of the Romberg table

$R(0,0) = 1.85914$			
$R(1,0) = 1.75393$	$R(1,1) =$ _____		
$R(2,0) = 1.72722$	$R(2,1) =$ _____	$R(2,2) =$ _____	
$R(3,0) = 1.72052$	$R(3,1) =$ _____	$R(3,2) =$ _____	$R(3,3) =$ _____

compute the remaining entries of the table.

Solution: Using Richardson extrapolation

$$R(1,1) = \frac{1}{3}[4R(1,0) - R(0,0)] = 1.71886$$
$$R(2,1) = \frac{1}{3}[4R(2,0) - R(1,0)] = 1.71832$$
$$R(3,1) = \frac{1}{3}[4R(3,0) - R(2,0)] = 1.7182842$$
$$R(2,2) = \frac{1}{15}[16R(2,1) - R(1,1)] = 1.7182827$$
$$R(3,2) = \frac{1}{15}[16R(3,1) - R(2,1)] = 1.7182818$$
$$R(3,3) = \frac{1}{63}[64R(3,2) - R(2,2)] = 1.7182818$$

(b) (2 marks) Explain how column 0 of the Romberg table is calculated.

Solution: Column 0 is computed using the recursive trapezoidal rule.

$$R(n,0) = \frac{1}{2}R(n-1,0) + \frac{h_n}{2}[2y_1 + 2y_3 + \cdots + 2y_{2^n-1}]$$

where $h_n = (b-a)/2^n$ and $R(0,0) = \frac{1}{2}[f(a) + f(b)]$.

Question 5, Lagrange Interpolation

(5 marks) Consider the following table of values for a function $f(x)$ on the interval $[0,3/2]$:

$x_0 = 0, \quad f(x_0) = 1/2$
$x_1 = 1/2, \quad f(x_1) = 1$
$x_2 = 1, \quad f(x_2) = 3$
$x_3 = 3/2, \quad f(x_3) = 13/4$

Write the degree 3 Lagrange interpolating polynomial for this table but do not simplify it.

Solution:

$$\ell_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = -\frac{4}{3}(x-x_1)(x-x_2)(x-x_3)$$

$$\ell_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = 4(x-x_0)(x-x_2)(x-x_3)$$

$$\ell_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = -4(x-x_0)(x-x_1)(x-x_3)$$

$$\ell_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{4}{3}(x-x_0)(x-x_1)(x-x_2)$$

$$P_3(x) = \ell_0(x)f(x_0) + \ell_1(x)f(x_1) + \ell_2(x)f(x_2) + \ell_3(x)f(x_3)$$

$$\begin{aligned} P_3(x) &= -\frac{2}{3}(x-1/2)(x-1)(x-3/2) + 4x(x-1)(x-3/2) \\ &\quad -12x(x-1/2)(x-3/2) + \frac{13}{3}x(x-1/2)(x-1) \end{aligned}$$