SOLUTIONS

MATH/COSC 3416 EA – TERM TEST 1

Tuesday, February 15, 2011, 8:30 am

Time Allowed: 75 minutes *Total Marks:* 20 *Instructor:* Barry G. Adams

Question 1

(a) (2 marks) The function $f(x) = \cos^2 x - \sin^2 x$ is a subtraction of nearly equal quantities for x near $\pi/4$ and other values that differ by 2π. This gives a loss of significant figures near $x = \pi/4$. Rewrite $f(x)$ to avoid this subtraction.

Solution: $\cos^2 x - \sin^2 x = \cos 2x$ and there are no subtractions.

(b) (2 marks) Consider the inverse cotangent series

$$\operatorname{arccot} x = \frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots = \frac{\pi}{2} - \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{2k+1} + \sum_{k=n+1}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

If $|x| < 1$ what value of n should be used so that the partial sum

$$S_n(x) = \frac{\pi}{2} - \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{2k+1}$$

is accurate to 0.5×10^{-3}?

Solution: Using the first term omitted we obtain, since $|x| < 1$,

$$\left| \frac{(-1)^{n+1} x^{2n+3}}{2n+3} \right| < \frac{1}{2n+3} < 0.5 \times 10^{-3}$$

Therefore $2n + 3 > 2 \times 10^3$ so we can choose any $n \geqslant 999$.

(c) (2 marks) It is known that the Bessel functions $J_n(x)$ satisfy the recurrence relation

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x), \quad n \geqslant 1.$$

Explain why this is not a useful recurrence relation for calculating values of $J_n(x)$ given values for $J_0(x)$ and $J_1(x)$ at a particular x.

Solution: In the computation of $J_{n+1}(x)$ any error in $J_n(x)$ is multiplied by $2n/x$ which rapidly increases as n increases for fixed x.

Question 2

(a) (2 marks) Consider the function $f(x) = x - e^{-x}$. Verify that this function changes sign on $[0,1]$ and use 3 steps of the bisection method to obtain an approximate root.

Solution: Since $f(0) < 0$ and $f(1) > 0$ then there is a root on the interval $[0,1]$. The first three iterations are

a	c	b
0^-	0.5^-	1^+
0.5^-	0.75^+	1^+
0.5^-	0.625^+	0.75^+

The approximation to the root is 0.625 after 3 steps.

(b) (2 marks) Write the root finding problem for $f(x) = x - e^{-x} = 0$ as a fixed point iteration $x = g(x)$ and calculate the first 3 iterations x_1, x_2, x_3 starting with $x_0 = 0.6$ and using 5 digit accuracy in your calculations.

Solution: $x = g(x)$ where $g(x) = e^{-x}$. The first three iterations beginning at $x_0 = 0.6$ are

$$x_1 = e^{-0.6} = 0.54881$$
$$x_2 = e^{-0.54881} = 0.57764$$
$$x_3 = e^{-0.57764} = 0.56122$$

Question 3

(a) (2 marks) Derive Newton's method for the root finding problem $f(x) = 0$ using the Taylor series expansion of $f(x)$ about x_0.

Solution: Start with the Taylor series expansion

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(\xi)(x - x_0)^2$$

and use the linear approximation (tangent line)

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

If the zero of $L(x)$ is at $x = x_1$ then

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0$$

Solving for x_1 gives

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In general we obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(b) (2 marks) Write the iteration formula, defining x_n in terms of x_{n-1}, for Newton's method for finding a root of the function $f(x) = x - e^{-x}$ given an initial guess x_0.

Solution: For $f(x) = x - e^{-x}$ we have $f'(x) = 1 + e^{-x}$ so Newton's method gives the iteration

$$x_{n+1} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}, \quad n \geqslant 0$$

(c) (4 marks) Write a Matlab function for Newton's method that uses a given tolerance and maximum number of iterations and has first line

```
function [x] = newton(f, fp, x0, tolerance, maxiter, printflag)
```

Solution:

```
function [x] = newton(f, fp, x0, tolerance, maxiter, printflag)

   x = x0;

   if printflag
      fprintf('%3s%24s%24s\n', 'n','x','err');
   end

   n = 1;
   while n <=  maxiter
      fpx = fp(x);

      if fpx == 0
         error('division by zero');
      end

      dx = f(x)/fpx;
      x = x - dx;
      err = abs(dx);

      if printflag
         fprintf('%3d%24.15e%24.15e\n', n,x,err);
```

```
      end

      if err < tolerance
         break;
      end

   n = n + 1;
   end

   if n > maxiter
      fprintf('Failure to converge in %d iterations\n', maxiter);
   end
   % x is automatically returned here
end
```

(d) (2 marks) Write the iteration formula, defining x_n in terms of x_{n-1} and x_{n-2}, for the secant method for finding a root of the function $f(x) = x - \cos x$ given two appropriate initial guesses x_0 and x_1.

Solution: The secant method for $f(x) = x - \cos x$ is

$$x_n = x_{n-1} - (x_{n-1} - \cos x_{n-1}) \left(\frac{x_{n-1} - x_{n-2}}{x_{n-1} - \cos x_{n-1} - x_{n-2} + \cos x_{n-2}} \right)$$