
MATH/COSC 3416 EL 01 FINAL EXAM

NUMERICAL METHODS I

Practice Exam

Time Allowed: 3 hours

Instructor: Barry G. Adams

1. Answer ALL questions in the booklets supplied.
 2. Any calculator is permitted.
 3. Total marks: 60.
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Question 1 (6 marks)

- (a) [2 marks] Consider the two recurrence relations

$$y_{n+1} = \frac{1}{n+1} - \frac{n}{2}y_n, \quad y_0 = 1$$

$$E_n = 1 - \frac{1}{n+1}E_{n-1}, \quad E_0 = 1$$

For each recurrence relation indicate whether it is numerically stable or not and give reasons for your answers.

- (b) [2 marks] Explain how accurate values of the function $f(x) = \cos x - 1$ can be computed near $x = 0$.
- (c) [2 marks] Write a pseudo-code algorithm to show how the polynomial $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n$ can be efficiently calculated at a given value of x .

Question 2 (6 marks)

- (a) [2 marks] Newton's method for finding a root of $f(x) = 0$ is based on the fixed point iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Derive this formula from the Taylor series expansion of $f(x)$.

- (b) [2 marks] Write Newton's method for $x^4 - 2x^2 + x - 3 = 0$ and use it to find a root accurate to at least 4 significant figures starting with initial guess $x_0 = 2.0$.
- (c) [2 marks] Apply 3 iterations of the bisection method to find the root of the equation $x - \cos x = 0$ that lies in the interval $[0.7, 0.8]$.

Question 3 (8 marks)

- (a) [3 marks] Consider the following table of values for a function
- $f(x)$
- on the interval
- $[0, 3/2]$
- :

$$\begin{aligned} x_0 &= 0, & f(x_0) &= 1/3 \\ x_1 &= 1/2, & f(x_1) &= 1 \\ x_2 &= 1, & f(x_2) &= 2 \\ x_3 &= 3/2, & f(x_3) &= 3 \end{aligned}$$

Write the degree 3 Lagrange interpolating polynomial for this table but do not simplify it.

- (b) [3 marks] The entries in the Newton divided difference table are constructed using the formula

$$F_{ij} = \frac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}}.$$

Complete the following Newton divided difference table that uses the values in Question 3(a)

$$\begin{array}{ccccccc} x_0 = 0 & F_{00} = 1/3 & & & & & \\ x_1 = 1/2 & F_{10} = 1 & F_{11} = xxxx & & & & \\ x_2 = 1 & F_{20} = 2 & F_{21} = xxxx & F_{22} = xxxx & & & \\ x_3 = 3/2 & F_{30} = 3 & F_{31} = xxxx & F_{32} = xxxx & F_{33} = xxxx & & \end{array}$$

- (c) [2 marks] Using this table write the degree 3 interpolating polynomial in Newton form and evaluate it at
- $x = 3/2$

Question 4 (4 marks)

Suppose a table is to be prepared for the function $f(x) = e^{-x}$ on the interval $0 \leq x \leq 1$. If h is the step size in x between table rows, what is an appropriate number of table rows so that linear interpolation will give an absolute error at most 0.5×10^{-3} ? Recall that the error formula for a linear interpolating polynomial $P(x)$ is

$$|f(x) - P(x)| \leq \frac{Mh^2}{8}, \quad \text{where } M = \max_{\xi \in [0,1]} |f''(\xi)|.$$

Question 5 (8 marks)

- (a) [3 marks] Using Taylor series expansions derive the
- $O(h^2)$
- central difference approximation

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

- (b) [5 marks] Using Richardson extrapolation and Taylor series expansions derive the
- $O(h^4)$
- derivative approximation

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$$

Question 6 (11 marks)

Consider the trapezoidal rule

$$\int_a^b f(x) dx = \frac{h}{2} [f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi), \quad \xi \in [a, b],$$

where $a = x_0$, $b = x_1$, $h = x_1 - x_0$.

- (a) [4 marks] Derive the composite trapezoidal rule

$$\int_a^b f(x) dx = T(f, h) - \frac{b-a}{12} h^2 f''(\mu), \quad \text{where } a \leq \mu \leq b \text{ and}$$

$$T(f, h) = \frac{h}{2} \left[f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right]$$

Here $x_0 = a$, $x_n = b$, $x_k = a + kh$, and $h = (b - a)/n$.

- (b) [3 marks] Evaluate the integral $\int_0^{2/10} \frac{dx}{x+2}$ using the composite trapezoidal rule with 5 steps (subintervals).
- (c) [1 mark] Use the error formula in (a) to estimate the error in your answer to (b)
- (d) [1 mark] For the integral in (b), how many steps (subintervals) would be needed to obtain an absolute error less than or equal to 0.5×10^{-4} .
- (e) [2 marks] Complete the following Romberg table to find an approximate value for the integral $\int_0^1 \frac{4}{1+x^2} dx$.

$$R(0, 0) = 3.0000$$

$$R(1, 0) = 3.1000 \quad R(1, 1) = x.xxxxxx$$

$$R(2, 0) = 3.1312 \quad R(2, 1) = x.xxxxxx \quad R(2, 2) = x.xxxxxx$$

$$R(3, 0) = 3.1390 \quad R(3, 1) = x.xxxxxx \quad R(3, 2) = x.xxxxxx \quad R(3, 3) = x.xxxxxx$$

Question 7 (12 marks)

- (a) [2 marks] Write a pseudo-code version of the naive gaussian algorithm that solves the linear system $\mathbf{Ax} = \mathbf{b}$. Write only the elimination part, not the back substitution part.
- (b) [2 marks] Explain some methods that try to improve on naive gaussian elimination by trying to avoid zero pivots, reduce round off error, or increase efficiency.
- (c) [2 marks] Given the LU decomposition of a matrix \mathbf{A} explain how it can be used to solve the linear system $\mathbf{Ax} = \mathbf{b}$
- (d) [2 marks] For algorithms that may require row exchanges show how your answer to (c) needs to be modified.

(e) [4 marks] Find the LU decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}$$

and use it to solve the linear system $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{b} = [8, 7, 14, -7]^T$

Question 8 (5 marks)

Consider the following first order differential equation and initial condition

$$x'(t) = x^2 + t^2 + 1, \quad x(1) = -3,$$

where $x'(t)$ denotes the derivative of x with respect to t .

- (a) [2 marks] Use Euler's method to find an approximation to $x(1.1)$ using a step size $h = 0.05$.
- (b) [3 marks] Develop the 4th order Taylor series method for solving this initial value problem (keep terms to order h^4) by calculating the k -th derivatives $x^{(k)}$, $k = 1, 2, 3, 4$. Express your answer as a pseudo-code algorithm that uses the given initial condition and calculates the solution at t_{end} , given the number of steps n .