MATH/COSC 3416 EL 01 FINAL EXAM
NUMERICAL METHODS I
Practice Exam

Time Allowed: 3 hours
Instructor: Barry G. Adams

1. Answer ALL questions in the booklets supplied.
2. Any calculator is permitted.
3. Total marks: 60.

Question 1  (6 marks)

(a) [2 marks] Consider the two recurrence relations

\[ y_{n+1} = \frac{1}{n+1} - \frac{n}{2}y_n, \quad y_0 = 1 \]
\[ E_n = 1 - \frac{1}{n+1}E_{n-1}, \quad E_0 = 1 \]

For each recurrence relation indicate whether it is numerically stable or not and give reasons for your answers.

(b) [2 marks] Explain how accurate values of the function \( f(x) = \cos x - 1 \) can be computed near \( x = 0 \).

(c) [2 marks] Write a pseudo-code algorithm to show how the polynomial \( p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n \) can be efficiently calculated at a given value of \( x \).

Question 2  (6 marks)

(a) [2 marks] Newton’s method for finding a root of \( f(x) = 0 \) is based on the fixed point iteration formula

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

Derive this formula from the Taylor series expansion of \( f(x) \).

(b) [2 marks] Write Newton’s method for \( x^4 - 2x^2 + x - 3 = 0 \) and use it to find a root accurate to at least 4 significant figures starting with initial guess \( x_0 = 2.0 \).

(c) [2 marks] Apply 3 iterations of the bisection method to find the root of the equation \( x - \cos x = 0 \) that lies in the interval \([0.7, 0.8] \).
Question 3  (8 marks)

(a) [3 marks] Consider the following table of values for a function \( f(x) \) on the interval \([0, 3/2]\):

\[
\begin{align*}
  x_0 &= 0, \quad f(x_0) = 1/3 \\
  x_1 &= 1/2, \quad f(x_1) = 1 \\
  x_2 &= 1, \quad f(x_2) = 2 \\
  x_3 &= 3/2, \quad f(x_3) = 3 
\end{align*}
\]

Write the degree 3 Lagrange interpolating polynomial for this table but do not simplify it.

(b) [3 marks] The entries in the Newton divided difference table are constructed using the formula

\[
F_{ij} = \frac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}}.
\]

Complete the following Newton divided difference table that uses the values in Question 3(a)

\[
\begin{array}{cccc}
  x_0 &= 0 & F_{00} = 1/3 \\
  x_1 &= 1/2 & F_{10} = 1 & F_{11} = xxxx \\
  x_2 &= 1 & F_{20} = 2 & F_{21} = xxxx & F_{22} = xxxx \\
  x_3 &= 3/2 & F_{30} = 3 & F_{31} = xxxx & F_{32} = xxxx & F_{33} = xxxx \\
\end{array}
\]

(c) [2 marks] Using this table write the degree 3 interpolating polynomial in Newton form and evaluate it at \( x = 3/2 \)

Question 4  (4 marks)

Suppose a table is to be prepared for the function \( f(x) = e^{-x} \) on the interval \(0 \leq x \leq 1\). If \( h \) is the step size in \( x \) between table rows, what is an appropriate number of table rows so that linear interpolation will give an absolute error at most \(0.5 \times 10^{-3}\)? Recall that the error formula for a linear interpolating polynomial \( P(x) \) is

\[
|f(x) - P(x)| \leq \frac{Mh^2}{8}, \quad \text{where} \quad M = \max_{\xi \in [0,1]} |f''(\xi)|.
\]

Question 5  (8 marks)

(a) [3 marks] Using Taylor series expansions derive the \( O(h^2) \) central difference approximation

\[
f'(x) = \frac{f(x+h) - f(x-h)}{2h}
\]

(b) [5 marks] Using Richardson extrapolation and Taylor series expansions derive the \( O(h^4) \) derivative approximation

\[
f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}
\]
Question 6  (11 marks)
Consider the trapezoidal rule
\[ \int_a^b f(x) \, dx = \frac{h}{2} [f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi), \quad \xi \in [a, b], \]
where \( a = x_0, b = x_1, h = x_1 - x_0. \)

(a) [4 marks] Derive the composite trapezoidal rule
\[ \int_a^b f(x) \, dx = T(f, h) - \frac{b-a}{12} h^2 f''(\mu), \quad \text{where} \ a \leq \mu \leq b \]
\[ T(f, h) = \frac{h}{2} \left[ f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right] \]
Here \( x_0 = a, x_n = b, x_k = a + kh, \) and \( h = (b-a)/n. \)

(b) [3 marks] Evaluate the integral \( \int_0^{2/10} \frac{dx}{x+2} \) using the composite trapezoidal rule with 5 steps (subintervals).

(c) [1 mark] Use the error formula in (a) to estimate the error in your answer to (b)

(d) [1 mark] For the integral in (b), how many steps (subintervals) would be needed to obtain an absolute error less than or equal to \( 0.5 \times 10^{-4}. \)

(e) [2 marks] Complete the following Romberg table to find an approximate value for the integral \( \int_0^1 \frac{4}{1+x} \, dx. \)

\[
\begin{array}{c|c|c}
R(0, 0) & 3.0000 & \\
R(1, 0) & 3.1000 & R(1, 1) = .xxxxx \\
R(2, 0) & 3.1312 & R(2, 1) = .xxxxx \quad R(2, 2) = .xxxxx \\
R(3, 0) & 3.1390 & R(3, 1) = .xxxxx \quad R(3, 2) = .xxxxx \quad R(3, 3) = .xxxxx \\
\end{array}
\]

Question 7  (12 marks)

(a) [2 marks] Write a pseudo-code version of the naive gaussian algorithm that solves the linear system \( Ax = b. \) Write only the elimination part, not the back substitution part.

(b) [2 marks] Explain some methods that try to improve on naive gaussian elimination by trying to avoid zero pivots, reduce round off error, or increase efficiency.

(c) [2 marks] Given the LU decomposition of a matrix \( A \) explain how it can be used to solve the linear system \( Ax = b \)

(d) [2 marks] For algorithms that may require row exchanges show how your answer to (c) needs to be modified.
(e) [4 marks] Find the LU decomposition of the matrix

\[ A = \begin{bmatrix}
1 & 1 & 0 & 3 \\
2 & 1 & -1 & 1 \\
3 & -1 & -1 & 2 \\
-1 & 2 & 3 & -1
\end{bmatrix} \]

and use it to solve the linear system \( Ax = b \) with \( b = [8, 7, 14, -7]^T \)

**Question 8** (5 marks)

Consider the following first order differential equation and initial condition

\[ x'(t) = x^2 + t^2 + 1, \quad x(1) = -3, \]

where \( x'(t) \) denotes the derivative of \( x \) with respect to \( t \).

(a) [2 marks] Use Euler’s method to find an approximation to \( x(1.1) \) using a step size \( h = 0.05 \).

(b) [3 marks] Develop the 4th order Taylor series method for solving this initial value problem (keep terms to order \( h^4 \)) by calculating the \( k \)-th derivatives \( x^{(k)}, k = 1, 2, 3, 4 \). Express your answer as a pseudo-code algorithm that uses the given initial condition and calculates the solution at \( t_{\text{end}} \), given the number of steps \( n \).