TOTAL: 40 MARKS

Assignment 4, MATH/COSC 3416, Numerical Methods I
Due Date: Mar. 12, 2010

Question 1 (Trapezoidal Rule)

6 **(a)** Page 200, Problems 5.2.2, 5.2.3, 5.2.5

4 **(b)** Page 203, Computer Problems 5.2.2 (a) and (b) only
Use our `trap` function for this problem and compute the values of the integral using $n = 10$, 100, 1000, 10000. Ignore reference to computer problems 5.2.1.

10 **Question 2 (Recursive Trapezoidal Rule)**

Approximate $\int_0^\pi \sin x\,dx$ using the recursive trapezoidal rule to compute $R(0,0)$ (step size $h = \pi$), $R(1,0)$ (step size $h = \pi/2$), $R(2,0)$ (step size $h = \pi/4$), and $R(3,0)$ (step size $h = \pi/8$). Do it by hand (NOT Matlab) keeping 6 significant figures in each calculation.

Question 3 (Simpson's Rule)

6 **(a)** Pages 227, Problem 6.1.2 (b), 6.1.4. NOTE: in 6.1.2(b) assume that $|f^{(4)}(\xi)| \leq 313.1$.

4 **(b)** Do Question 1(b) but now use our `simp` function for n=10, 100, 1000, 10000.

Question 4 (Romberg Algorithm)

6 **(a)** Pages 212–213, Problems 5.3.1, 5.3.5

4 **(b)** Pages 214–215, Computer Problems 5.3.1, 5.3.9
Use our `romberg` procedure to do these problems.

ASSIGNMENT #4 SOLUTIONS

1(a) [PROBLEM 5.2.2] Compute an approximate value of $\int_0^1 (x^2+1)^{-1}\,dx$ by using the composite trapezoidal rule with 3 points. Then compare the actual value of the integral. Next determine the error formula and numerically verify an upper bound on it

ANSWER ↓

Exact answer is $\int_0^1 \frac{dx}{x^2+1} = \arctan(1) = 0.7854\ldots$

For trapezoidal rule the nodes are $x_0 = 0.0$, $x_1 = 0.5$, $x_2 = 1.0$ (see Equation 1, page 191). The trapezoidal approximation is

$$T(f;P) = \frac{h}{2}\left[f(0) + f(1)\right] + h\,f\left(\tfrac{1}{2}\right) = \frac{1}{4}\left(1 + \frac{1}{2}\right) + \frac{1}{2}\cdot\frac{1}{1/4+1}$$

$= 31/40 = 0.775$

The error formula using $h = 1/2$ is

$$-\frac{1}{12}(b-a)h^2 f''(\xi) = -\frac{1}{48} f''(\xi), \quad 0 \le \xi \le 1$$

Calculate $f''(x) = (6x^2-2)/(x^2+1)^3$. Graphically plot $f''(x)$ on $[0,1]$ to arrive at a maximum of 2 (taking absolute values)

This gives error estimate $|0.7854 - 0.775|$

$= 0.0104 \le \frac{1}{24} = 0.0417$

1(a) [PROBLEM 5.2.3] (Continuation) Having computed $R(1,0)$ in the preceding problem, compute $R(2,0)$ by using formula $R(n,0) = \frac{1}{2} R(n-1,0) + h \sum_{k=1}^{2^{n-1}} f[a + (2k-1)h]$ using $h = \frac{b-a}{2^n}$ and $R(0,0) = \frac{1}{2}(b-a)[f(a)+f(b)]$

ANSWER ↓

Using the nodes $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$ with $h = 1/2$ we obtain $R(1,0) = 31/40$ [from previous question]

$R(2,0) = \frac{1}{2} R(1,0) + h[f(1/4) + f(3/4)]$ where we have used the new nodes at 1/4 and 3/4. Now $h = 1/4$ so

$$R(2,0) = \frac{31}{80} + \frac{1}{4}\left[\frac{1}{\frac{1}{16}+1} + \frac{1}{\frac{9}{16}+1}\right] = 0.7828$$

The exact value is 0.7854

1(a) [PROBLEM 5.2.5] If the composite trapezoidal rule is used to compute $\int_{-1}^{2} \sin x \, dx$ with $h = 0.01$ give a realistic bound on the error.

ANSWER

$|\text{Error}| < \frac{1}{12}(b-a) h^2 |f''(\xi)| = \frac{1}{4}\left(\frac{1}{100}\right)^2 |f''(\xi)|$
$<= 0.25 \times 10^{-4}$ where we have used $f''(x) = -\sin x$ which implies that $|f''(\xi)| \leq 1$

1(b) [COMPUTER PROBLEM 5.2.2(a),(b)]

Use our trap function for this problem and compute the values of the integrals in (a),(b) using $n = 10, 100, 1000, 10000$ [ignore reference to problem 5.2.1.

SEE M-FILE question1b.m

QUESTION 2 Approximate $\int_0^{\pi} \sin x \, dx$ using recursive trapezoidal rule to compute $R(0,0)$, $R(1,0)$, $R(2,0)$ and $R(3,0)$.

Starting with nodes $x_0 = 0$, $x_1 = \pi$, $h = \pi$ we get

$$R(0,0) = \frac{1}{2}(b-a)(f(a)+f(b)) = \frac{\pi}{2}(0) = \boxed{0}$$

Now subdivide and use nodes $x_0 = 0$, $x_1 = \pi/2$, $x_2 = \pi$ with $h = \pi/2$ to get

$$R(1,0) = \frac{1}{2}R(0,0) + \frac{\pi}{2}\sin\frac{\pi}{2} = 0 + \frac{\pi}{2} \simeq \boxed{1.57080}$$

Now subdivide again and use nodes $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/2$, $x_3 = 3\pi/4$, $x_4 = \pi$ and $h = \pi/4$ to get

$$R(2,0) = \frac{1}{2}R(1,0) + \frac{\pi}{4}\left[\sin\frac{\pi}{4} + \sin\frac{3\pi}{4}\right] = \frac{\pi}{4} + \frac{\pi}{4}\left[\sin\frac{\pi}{4} + \sin\frac{3\pi}{4}\right]$$

$$\simeq \boxed{1.89612}$$

Now subdivide again and use nodes $x_0 = 0$, $x_1 = \pi/8$, $x_2 = \pi/4$, $x_3 = 3\pi/8$, $x_4 = \pi/2$, $x_5 = 5\pi/8$, $x_6 = 3\pi/4$, $x_7 = 7\pi/8$, $x_8 = \pi$ and $h = \pi/8$ to get

$$R(3,0) = \frac{1}{2}R(2,0) + \frac{\pi}{8}\left(\sin\frac{\pi}{8} + \sin\frac{3\pi}{8} + \sin\frac{5\pi}{8} + \sin\frac{7\pi}{8}\right)$$

$$\simeq \boxed{1.97423}$$

QUESTION 3a [PROBLEM 6.1.2 (b)]

Consider the integral $\int_0^1 \sin(\pi x^2/2)\, dx$. Suppose we want to integrate numerically with an error $< 10^{-3}$ what width h is needed if we wish to use the composite Simpsons Rule

Answer

$$|\text{Error}| = \frac{1}{180}(b-a)h^4 |f^{(4)}(\xi)|$$

Assume that $|f^{(4)}(\xi)| \leq 313.1$ [this can be shown using Maple and would be complicated to do by hand.

$$\therefore |\text{Error}| \leq \frac{1}{180} h^4 (313.1) < 10^{-3} \quad \{ b-a = 1 \text{ here} \}$$

Solving for h gives $h^4 < 5.75 \times 10^{-4} \Rightarrow$ $\boxed{h < 0.15}$

so a good stepsize would be $\boxed{h = 0.125}$ $(1/8)$

QUESTION 3(a) [PROBLEM 6.1.4] Find the approximate value of $\int_1^2 x^{-1} dx$ using composite simpsons rule with $h = 0.25$. Give a bound on the error

ANSWER: $\int_1^2 \frac{dx}{x} = \frac{h}{3}\left[f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)\right]$

$$= \frac{0.25}{3}\left[1 + \frac{4}{1.25} + \frac{2}{1.5} + \frac{4}{1.75} + \frac{1}{2}\right] \simeq 0.6933$$

$$|\text{Error}| < \frac{1}{180}(b-a)h^4|f^{(4)}(\xi)| < \frac{1}{180}\left(\frac{1}{4}\right)^4(24) \simeq 5.2 \times 10^{-4}$$

where we have used $f^{(4)}(x) = 24x^{-6}$ and $|f^{(4)}(x)| \leq 24$ on the interval $[1,2]$. The exact value of the integral is $\ln 2 \simeq 0.693147$ so the actual error is $|\ln 2 - 0.6933| = 1.5 \times 10^{-4}$ which is smaller than our estimate

FOR QUESTION 3(b) SEE question3b.m

QUESTION 4(a) [PROBLEM 5.3.1] What is $R(5,3)$ if $R(5,2) = 12$ and $R(4,2) = -51$ in the Romberg algorithm

Answer

$$R(5,3) = \frac{64R(5,2) - R(4,2)}{63} = \frac{64(12)+51}{63} = \frac{819}{63} = 13$$

QUESTION 4(a) [PROBLEM 5.3.5] By the Romberg algorithm approximate $\int_0^2 4dx/(1+x^2)$ by evaluating $R(1,1)$

Answer

To approximate the integral first use step size $h=2$ to obtain

$$R(0,0) = \frac{1}{2}(b-a)[f(a)+f(b)] = f(0)+f(2) = 24/5$$

Now subdivide and use the nodes $x_0=0$, $x_1=1$, $x_2=2$, $h=1$ to obtain

$$R(1,0) = \frac{1}{2}R(0,0) + 1(f(1)) = 12/5 + 2 = 22/5$$

Now use Richardson extrapolation to obtain

$$R(1,1) = \frac{4R(1,0) - R(0,0)}{3} = \frac{1}{3}\left(\frac{88}{5} - \frac{24}{5}\right) = \frac{64}{15} \simeq 4.267$$

QUESTION 4(b) [COMPUTER PROBLEM 5.3.1] Compute 8 rows and columns in the Romberg array for $\int_{1.3}^{2.19} x^{-1}\sin x\,dx$

See question4b.m

QUESTION 4(b) [COMPUTER PROBLEM 5.3.9] Calculate $\int_0^1 \frac{\sin x}{\sqrt{x}}dx$ by the Romberg algorithm: The substitution $u=\sqrt{x}$ gives $\int_0^1 \frac{\sin x}{\sqrt{x}}dx = \int_0^1 2\sin(u^2)\,du$ to which Romberg can be applied

SEE QUESTION4b.m

```
% question1b.m

for n = [10, 100, 1000, 10000]
   trap(@(x) sin(x), 0, pi, n)
end

for n = [10, 100, 1000, 10000]
   trap(@(x) exp(x), 0, 1, n)
end

%>> question1b
%ans =
%   1.98352353750945
%ans =
%   1.99983550388744
%ans =
%   1.99999835506566
%ans =
%   1.99999998355066
%ans =
%   1.71971349138931
%ans =
%   1.71829614745042
%ans =
%   1.71828197164920
%ans =
%   1.71828182989095
%>>
%
% Exact values is 2 for integral of sin(x)
% Exact value  is e-1 = 1.718281828 for
% integral of exp(x).
```

```
% question3b.m

for n = [10, 100, 1000, 10000]
   simp(@(x) sin(x), 0, pi, n)
end

for n = [10, 100, 1000, 10000]
   simp(@(x) exp(x), 0, 1, n)
end

%>> question3b
%ans =
%   2.00010951731500
%ans =
%   2.00000001082450
%ans =
%   2.00000000000108
%ans =
%   2.00000000000000
%ans =
%   1.71828278192482
%ans =
%   1.71828182855450
%ans =
%   1.71828182845906
%ans =
%   1.71828182845904
%>>
%
% Exact values is 2 for integral of sin(x)
% Exact value  is e-1 = 1.718281828 for
% integral of exp(x).
```

```
% question4b.m

f = @(x) sin(x)/x
romberg(f,1.3, 2.19, 8)

f = @(u) 2*sin(u*u)
romberg(f, 0, 1, 8)
```

```
>> question4b
f =
    @(x) sin(x)/x
ans =
  Columns 1 through 3
   0.49530447343402                  0                  0
   0.49880688930716   0.49997436126487                  0
   0.49967949805781   0.49997036764136   0.49997010139980
   0.49989746398287   0.49997011929122   0.49997010273454
   0.49995194383739   0.49997010378889   0.49997010275540
   0.49996556307457   0.49997010282030   0.49997010275573
   0.49996896783847   0.49997010275977   0.49997010275574
   0.49996981902661   0.49997010275599   0.49997010275574
  Columns 4 through 6
                  0                  0                  0
                  0                  0                  0
                  0                  0                  0
   0.49997010275573                  0                  0
   0.49997010275574   0.49997010275574                  0
   0.49997010275574   0.49997010275574   0.49997010275574
   0.49997010275574   0.49997010275574   0.49997010275574
   0.49997010275574   0.49997010275574   0.49997010275574
  Columns 7 through 8
                  0                  0
                  0                  0
                  0                  0
                  0                  0
                  0                  0
                  0                  0
   0.49997010275574                  0
   0.49997010275574   0.49997010275574
>> question4b
f =
    @(x) sin(x)/x
ans =
  Columns 1 through 3
   0.49530447343402                  0                  0
   0.49880688930716   0.49997436126487                  0
   0.49967949805781   0.49997036764136   0.49997010139980
   0.49989746398287   0.49997011929122   0.49997010273454
   0.49995194383739   0.49997010378889   0.49997010275540
   0.49996556307457   0.49997010282030   0.49997010275573
   0.49996896783847   0.49997010275977   0.49997010275574
   0.49996981902661   0.49997010275599   0.49997010275574
  Columns 4 through 6
                  0                  0                  0
                  0                  0                  0
                  0                  0                  0
   0.49997010275573                  0                  0
```

```
   0.49997010275574   0.49997010275574                  0
   0.49997010275574   0.49997010275574   0.49997010275574
   0.49997010275574   0.49997010275574   0.49997010275574
   0.49997010275574   0.49997010275574   0.49997010275574
  Columns 7 through 8
                  0                  0
                  0                  0
                  0                  0
                  0                  0
                  0                  0
                  0                  0
   0.49997010275574                  0
   0.49997010275574   0.49997010275574
f =
    @(u) 2*sin(u*u)
ans =
  Columns 1 through 3
   0.84147098480790                  0                  0
   0.66813945165847   0.61036227394200                  0
   0.63195072151844   0.61988781147176   0.62052284730707
   0.62336047896188   0.62049706477636   0.62053768166334
   0.62124073361898   0.62053415183801   0.62053662430878
   0.62071252130912   0.62053645053917   0.62053660378592
   0.62058057574857   0.62053659389505   0.62053660345211
   0.62054759607454   0.62053660284986   0.62053660344685
  Columns 4 through 6
                  0                  0                  0
                  0                  0                  0
                  0                  0                  0
   0.62053791712931                  0                  0
   0.62053660752538   0.62053660238968                  0
   0.62053660346016   0.62053660344421   0.62053660344524
   0.62053660344681   0.62053660344676   0.62053660344676
   0.62053660344676   0.62053660344676   0.62053660344676
  Columns 7 through 8
                  0                  0
                  0                  0
                  0                  0
                  0                  0
                  0                  0
                  0                  0
   0.62053660344676                  0
   0.62053660344676   0.62053660344676
>> >> question4b
f =
    @(x) sin(x)/x
ans =
  Columns 1 through 3
   0.49530447343402                  0                  0
```

```
   0.49880688930716   0.49997436126487                  0
   0.49967949805781   0.49997036764136   0.49997010139980
   0.49989746398287   0.49997011929122   0.49997010273454
   0.49995194383739   0.49997010378889   0.49997010275540
   0.49996556307457   0.49997010282030   0.49997010275573
   0.49996896783847   0.49997010275977   0.49997010275574
   0.49996981902661   0.49997010275599   0.49997010275574
  Columns 4 through 6
                  0                  0                  0
                  0                  0                  0
                  0                  0                  0
   0.49997010275573                  0                  0
   0.49997010275574   0.49997010275574                  0
   0.49997010275574   0.49997010275574   0.49997010275574
   0.49997010275574   0.49997010275574   0.49997010275574
   0.49997010275574   0.49997010275574   0.49997010275574
  Columns 7 through 8
                  0                  0
                  0                  0
                  0                  0
                  0                  0
                  0                  0
                  0                  0
   0.49997010275574                  0
   0.49997010275574   0.49997010275574
>> question4b
f =
    @(x) sin(x)/x
ans =
  Columns 1 through 3
   0.49530447343402                  0                  0
   0.49880688930716   0.49997436126487                  0
   0.49967949805781   0.49997036764136   0.49997010139980
   0.49989746398287   0.49997011929122   0.49997010273454
   0.49995194383739   0.49997010378889   0.49997010275540
   0.49996556307457   0.49997010282030   0.49997010275573
   0.49996896783847   0.49997010275977   0.49997010275574
   0.49996981902661   0.49997010275599   0.49997010275574
  Columns 4 through 6
                  0                  0                  0
                  0                  0                  0
                  0                  0                  0
   0.49997010275573                  0                  0
   0.49997010275574   0.49997010275574                  0
   0.49997010275574   0.49997010275574   0.49997010275574
   0.49997010275574   0.49997010275574   0.49997010275574
   0.49997010275574   0.49997010275574   0.49997010275574
  Columns 7 through 8
                  0                  0
```

```
                   0                   0
                   0                   0
                   0                   0
                   0                   0
                   0                   0
    0.49997010275574                   0
    0.49997010275574    0.49997010275574
f =
    @(u) 2*sin(u*u)
ans =
  Columns 1 through 3
    0.84147098480790                   0                   0
    0.66813945165847    0.61036227394200                   0
    0.63195072151844    0.61988781147176    0.62052284730707
    0.62336047896188    0.62049706477636    0.62053768166334
    0.62124073361898    0.62053415183801    0.62053662430878
    0.62071252130912    0.62053645053917    0.62053660378592
    0.62058057574857    0.62053659389505    0.62053660345211
    0.62054759607454    0.62053660284986    0.62053660344685
  Columns 4 through 6
                   0                   0                   0
                   0                   0                   0
                   0                   0                   0
    0.62053791712931                   0                   0
    0.62053660752538    0.62053660238968                   0
    0.62053660346016    0.62053660344421    0.62053660344524
    0.62053660344681    0.62053660344676    0.62053660344676
    0.62053660344676    0.62053660344676    0.62053660344676
  Columns 7 through 8
                   0                   0
                   0                   0
                   0                   0
                   0                   0
                   0                   0
                   0                   0
    0.62053660344676                   0
    0.62053660344676    0.62053660344676
>>
```