NUMERICAL METHODS

SOLUTIONS TO ASSIGNMENT 3

QUESTION 1(a) (problem 4.1.1) { Find, using Lagrange method, the interpolating polynomial

x	0	2	3	4
y	7	11	28	63

. first calculate $l_0, \dots, l_3$

$$l_0(x) = \frac{(x-2)(x-3)(x-4)}{(0-2)(0-3)(0-4)} = -\frac{1}{24}(x-2)(x-3)(x-4)$$

$$l_1(x) = \frac{(x-0)(x-3)(x-4)}{(2-0)(2-3)(2-4)} = \frac{1}{4}(x-0)(x-3)(x-4)$$

$$l_2(x) = \frac{(x-0)(x-2)(x-4)}{(3-0)(3-2)(3-4)} = -\frac{1}{3}(x-0)(x-2)(x-4)$$

$$l_3(x) = \frac{(x-0)(x-2)(x-3)}{(4-0)(4-2)(4-3)} = \frac{1}{8}(x-0)(x-2)(x-3)$$

Therefore the Lagrange interpolating polynomial is

$$p_3(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x) + y_3 l_3(x)$$

$$= -\frac{7}{24}(x-2)(x-3)(x-4) + \frac{11}{4}(x-0)(x-3)(x-4)$$

$$- \frac{28}{3}(x-0)(x-2)(x-3) + \frac{63}{8}(x-0)(x-2)(x-3)$$

QUESTION 1(b) (using Newton's form)

$\boxed{p_0(x) = 7}$ using first node (x_0)

$p_1(x) = 7 + c(x-0)$:

Substitute $p_1(2) = 11$ to get $7 + 2c = 11 \Rightarrow c = 2$

$\boxed{p_1(x) = 7 + 2x}$.

$p_2(x) = p_1(x) + c(x-0)(x-2) = 7 + 2x + c(x-0)(x-2)$

Substitute $p_2(3) = 28$ to get $7 + 6 + 3c = 28 \Rightarrow c = 5$

$\boxed{p_2(x) = 7 + 2x + 5(x-0)(x-2)}$

$p_3(x) = p_2(x) + c(x-0)(x-2)(x-3) =$

$= 7 + 2x + 5(x-0)(x-2) + c(x-0)(x-2)(x-3)$

Substitute $p_3(4) = 63$ to get $7 + 8 + 40 + 8c = 63 \Rightarrow c = 1$

$\boxed{p_3(x) = 7 + 2x + 5x(x-2) + x(x-2)(x-3)}$

QUESTION 1(c)

we need to interpolate at 11 equally spaced points on the interval $[1,6]$. Distance between these points is $\frac{6-1}{11-1} = 0.5$. Therefore the points are

$$x_0=1,\ x_1=1.5,\ x_2=2,\ \ldots,\ x_9=5.5,\ x_{10}=6$$

Then we calculate the newton form of the interpolating polynomial. In Matlab.

```
xp = 1:0.5:6
yp = atan(xp)
a  = newton-poly(xp,yp)
```

Then we evaluate this polynomial at 33 equally spaced points on interval $[0,8]$. Distance between points is $\frac{8-0}{33-1} = 0.25$. These points can be expressed as $t_k = 0.25k$ for $k=0$ to 32

see question 1c.m for program

Conclusions: no errors at nodes as expected. error is smallest ($\simeq 10^{-7}$) near center of interval $[0,8]$. As we get closer to endpoints error quickly increases. This happens because on $[0,8]$ we are going outside the interpolation interval $[1,6]$.

```
% question 1(c) assignment 3

% First use 11 points to define the degree 10
% interpolating polynomial for arc tangent

xp = 1:0.5:6
yp = atan(xp);
a = newton_poly(xp,yp);

% Interpolate at 33 points and display results

for k = 0 : 32
   t = 0.25 * k;
   val = newton_eval(xp, a, t);
   err = abs(atan(t) - val);
   fprintf('%4d%8.3f%16.10f%20.10e\n', k,t,val,err);
end
```

```
>> question1c
xp =
  Columns 1 through 3
   1.00000000000000   1.50000000000000   2.00000000000000
  Columns 4 through 6
   2.50000000000000   3.00000000000000   3.50000000000000
  Columns 7 through 9
   4.00000000000000   4.50000000000000   5.00000000000000
  Columns 10 through 11
   5.50000000000000   6.00000000000000
   0    0.000    -0.1053163147    1.0531631467e-001
   1    0.250     0.2130806365    3.1898026585e-002
   2    0.500     0.4563962815    7.2513275070e-003
   3    0.750     0.6424669733    1.0341355095e-003
   4    1.000     0.7853981634    0.0000000000e+000
   5    1.250     0.8960899620    3.4577406591e-005
   6    1.500     0.9827937232    0.0000000000e+000
   7    1.750     1.0516462667    3.9458668386e-006
   8    2.000     1.1071487178    0.0000000000e+000
   9    2.250     1.1525728741    8.7692262674e-007
  10    2.500     1.1902899497    0.0000000000e+000
  11    2.750     1.2220250013    3.2187400989e-007
  12    3.000     1.2490457724    0.0000000000e+000
  13    3.250     1.2722975763    1.8111034983e-007
  14    3.500     1.2924966678    0.0000000000e+000
  15    3.750     1.3101937839    1.5112254870e-007
  16    4.000     1.3258176637    0.0000000000e+000
  17    4.250     1.3397058456    1.8604753449e-007
  18    4.500     1.3521273809    0.0000000000e+000
  19    4.750     1.3632997547    3.4570941354e-007
  20    5.000     1.3734007669    2.2204460493e-016
  21    5.250     1.3825758554    1.0338813856e-006
  22    5.500     1.3909428270    0.0000000000e+000
  23    5.750     1.3985996839    5.8284098725e-006
  24    6.000     1.4056476494    2.2204460493e-016
  25    6.250     1.4122511123    1.1004772130e-004
  26    6.500     1.4187694760    6.2247757117e-004
  27    6.750     1.4260133110    2.2953396366e-003
  28    7.000     1.4356992391    6.7999669358e-003
  29    7.250     1.4512050950    1.7474942527e-002
  30    7.500     1.4787596104    4.0514815889e-002
  31    7.750     1.5292396036    8.6766504500e-002
  32    8.000     1.6207929289    1.7435159669e-001
>>
```

QUESTION 2(a) 4.1.7(a)

x	$f[\,]$	$f[-,-]$	$f[-,-,-]$	$f[-,-,-,-]$
-1	2			
1	-4	-3		
3	6	5	2	
5	10	2	-3/4	-11/24

given

second differences are $\frac{-4-2}{1-(-1)} = -3$, $\frac{6-(-4)}{3-1} = 5$, $\frac{10-6}{5-3} = 2$

third differences are $\frac{5-(-3)}{3-(-1)} = 2$, $\frac{2-5}{5-1} = -\frac{3}{4}$

4th difference $\frac{-3/4-2}{5-(-1)} = -11/24$

QUESTION 2(b) 4.1.7(b)

x	$f[\cdot]$	$f[\cdot\,\cdot]$	$f[\cdots]$	$f[\cdots]$
-1	2			
1	-4	-3		
3	46	25	7	
4	99.5	53.5	9.5	0.5

Second differences $\frac{-4-2}{1-(-1)} = -3$, $\frac{46-(-4)}{3-1} = 25$, $\frac{99.5-46}{4-3} = 53.5$

third differences $\frac{25-(-3)}{3-(-1)} = 7$, $\frac{53.5-25}{4-1} = 9.5$

4th differences $\frac{9.5-7}{4-(-1)} = 0.5$

QUESTION 3(a)

$f(x) = \tan x$ on $[0,1]$. Step size is determined using $\frac{1}{8} Mh^2 < 0.5 \times 10^{-6}$ where $M = \max_{x \in [0,1]} f''(x)$

Differentiating $\tan x$ twice gives $f''(x) = 2\sec^2 x \tan x$ which is an increasing function on $[0,1]$. Therefore $M = 2\sec^2 1 \cdot \tan 1 = 10.6698$ so $h^2 < \left(\frac{4}{M}\right) \times 10^{-6}$.

Therefore $h < (2/\sqrt{M}) \times 10^{-3} \simeq 0.0006122$.

Finally we get $n = \lceil 1/h \rceil = 1634$ so our table needs at least 1634 entries

QUESTION 3(b)

Now we use $\frac{1}{9\sqrt{3}} Mh^3 < 0.5 \times 10^{-6}$, $M = \max_{[0,1]} f'''(x)$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$ which is an increasing function on $[0,1]$. Therefore $M = 4\sec^2 1 \cdot \tan^2 1 + 2\sec^4 1$
$= 56.703$ and $h^3 < (4.5\sqrt{3}/M) \times 10^{-6}$ so

$h < 0.00516 \Rightarrow n = \lceil 1/h \rceil = 194$ so our table needs at least 194 entries

QUESTION 3(c) Problem 4.2.6

Given a table of $\sin x$ to 10 decimal places for x in $[0,2]$ with $h = 0.01$ a bound on the error is $\frac{1}{8} h^2 M$ where M is the maximum value of $\sin x$ on $[0,2]$.
Since $|\sin x| \le 1$ we obtain the error bound
$0.125 \times (0.01)^2 = 0.125 \times 10^{-4}$

QUESTION 3(c) Problem 4.2.7

Given the data $\sin(0.70) \approx 0.6442176872$
$\sin(0.71) \approx 0.6518337710$, $\cos(0.70) \approx 0.7648421872$
$\cos(0.71) \approx 0.7583618759$, find approximate values of $\sin(0.705)$ and $\cos(0.702)$ by linear interpolation.
What is the error?

$$p(x) = y_0 + \frac{x - x_0}{x_1 - x_0}(y_1 - y_0)$$ is best for hand calculations

The ratio $(x-x_0)/(x_1-x_0)$ is the fractional distance of x from x_0 to x_1

$$\sin(0.705) = 0.6442176872 + \tfrac{1}{2}(0.6518337710 - 0.6442176872)$$
$$= 0.6480257291$$

Exact value is $\sin(0.705) = 0.6480338295$ so absolute error is 0.81004×10^{-5}

$$\cos(0.702) = 0.7648421872 + \tfrac{2}{5}(0.7583618759 - 0.7648421872)$$
$$= 0.7635461250$$

Exact value is $\cos(0.702) = 0.7635522231$ so absolute error is 0.60981×10^{-5}

QUESTION 4(a)

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + O(h^3)$$
$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) + O(h^3)$$

$$f(x+h) - f(x-h) = 2h f'(x) + O(h^3)$$

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2)$$

$$\boxed{\text{Therefore } f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)}$$

QUESTION 4(b)

define $\phi(h) = [f(x+h) - f(x)]/2h$. Use the extrapolation formula for derivatives:

$$f'(x) = \phi(h) + a_2 h^2 + a_4 h^4 + a_6 h^6 + \cdots$$ to obtain

$$\phi(h) = f'(x) - a_2 h^2 - a_4 h^4 - a_6 h^6 - \cdots$$ ← multiply by 4 and
$$\phi(2h) = f'(x) - a_2 4h^2 - a_4 16h^4 - a_6 64h^6 \ldots$$ subtract

$$4\phi(h) - \phi(2h) = 3f'(x) + O(h^4).$$ Therefore

$$f'(x) \approx \frac{1}{3}[4\phi(h) - \phi(2h)] + O(h^4)$$
$$= \frac{4}{3}\left[\frac{f(x+h) - f(x-h)}{2h}\right] - \frac{1}{3}\left[\frac{f(x+2h) - f(x-2h)}{4h}\right] + O(h^4)$$

$$\boxed{f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + O(h^4)}$$

QUESTION 5(a)

For $x = 0.5$, $h = 1$, $f(x) = \tan x$ using 6 significant figures the first column of the richardson table is

$$D(0,0) = \phi(1) = \frac{1}{2}\left[\tan(1.5) - \tan(-0.5)\right] = 7.32386$$

$$D(1,0) = \phi(0.5) = 1\left[\tan(1.0) - \tan(0.0)\right] = 1.55741$$

$$D(2,0) = \phi(0.25) = \frac{1}{0.5}\left[\tan(0.75) - \tan(0.25)\right] = 1.35251$$

$$D(3,0) = \phi(0.125) = \frac{1}{0.25}\left[\tan(0.625) - \tan(0.375)\right] = 1.31143$$

Remaining columns are

$$D(1,1) = \frac{1}{3}\left[4D(1,0) - D(0,0)\right] = \frac{1}{3}\left[4(1.55741) - 7.32386\right] = -0.364740$$

$$D(2,1) = \frac{1}{3}\left[4D(2,0) - D(1,0)\right] = \frac{1}{3}\left[4(1.35251) - 1.55741\right] = 1.28421$$

$$D(3,1) = \frac{1}{3}\left[4D(3,0) - D(2,0)\right] = \frac{1}{3}\left[4(1.31143) - 1.35251\right] = 1.29774$$

$$D(2,2) = \frac{1}{15}\left[16D(2,1) - D(1,1)\right] = \frac{1}{15}\left[16(1.28421) + 0.364740\right] = 1.39414$$

$$D(3,2) = \frac{1}{15}\left[16D(3,1) - D(2,1)\right] = \frac{1}{15}\left[16(1.29774) - 1.28421\right] = 1.29864$$

$$D(3,3) = \frac{1}{63}\left[64D(3,2) - D(2,2)\right] = \frac{1}{63}\left[64(1.29864) - 1.39414\right] = 1.29712$$

Richardson table is

7.32386			
1.55741	-0.364740		
1.35251	1.28421	1.39414	
1.31143	1.29774	1.29864	1.29712

```
>> richardson(@tan, 0.5, 6, 1)
ans =
    7.3239         0         0         0         0         0
    1.5574   -0.3647         0         0         0         0
    1.3525    1.2842    1.3941         0         0         0
    1.3114    1.2977    1.2986    1.2971         0         0
    1.3017    1.2984    1.2984    1.2984    1.2985         0
    1.2992    1.2984    1.2984    1.2984    1.2984    1.2984
>> format long
>> richardson(@tan, 0.5, 6, 1)
ans =
  Columns 1 through 3
   7.32386121850776                  0                  0
   1.55740772465490  -0.36474343996272                  0
   1.35250907744607   1.28420952837646   1.39413972626574
   1.31143146026109   1.29773892119943   1.29864088072096
   1.30166118706794   1.29840442933690   1.29844879654606
   1.29924816152341   1.29844381967524   1.29844644569779
  Columns 4 through 6
                  0                  0                  0
                  0                  0                  0
                  0                  0                  0
   1.29712502602977                  0                  0
   1.29844574759090   1.29845092689114                  0
   1.29844640838274   1.29844641097408   1.29844640655970
>>
```