Answer 1 (the derivative experiment)

Solution  See the MATLAB m-files deriv.m, deriv2.m, and deriv3.m. Here is the MATLAB code for question 1(b).

\begin{verbatim}
 n = input('Enter number of iterations '); 

 exact = exp(0.5); 
 fprintf('Exact value = %.10e
', exact); 
 fprintf('%5s %18s %18s %18s
','n','h','r','err'); 
 h = 1.0; 
 for n = 1:n 
   r = (exp(0.5+h) - exp(0.5)) / h; 
   err = exact - r; 
   fprintf('%5d %18.10e %18.10e %18.10e
',n,h,r,err); 
   h = h / 4.0; 
 end
\end{verbatim}

In all three cases the problems arise from using the limit definition

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

to obtain approximations to the derivative: as \( h \) approaches 0 there will be a positive but nonzero value of \( h \) such that \( x + h \) is numerically equal to \( x \). Then \( r \) will be zero.
Question 2 (the inverse tangent function)

Solution to 2(a) The following calculations give the Taylor series for \( \arctan x \).

\[
\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k \quad \text{which converges for } -1 < x < 1
\]

\[
\frac{1}{1+t^2} = \sum_{k=0}^{\infty} (-t^2)^k = \sum_{k=0}^{\infty} (-1)^k t^{2k}
\]

\[
\arctan(x) = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^{2k} dt = \sum_{k=0}^{\infty} (-1)^k \int_0^x t^{2k} dt
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{2k-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\]

which converges for \(-1 < x \leq 1\).

Solution to 2(b) Since the series for \( 4 \arctan x \) is alternating the error committed in truncating it is less than the first term omitted. Let the absolute value this term be \( 4x^{2n-1}/(2n-1) \). Then for \( x = 1 \) we require that \( 4/(2n-1) < 1/10^2 \). This gives \( n = 201 \) for the first term omitted. Since the powers in the series are odd numbers then \( n = 199 \) is the last term so we can use the Taylor polynomial of degree 199.

Solution to 2(c) Since the series converges only on the interval \(-1 < x \leq 1\) we expect that near the boundary of this interval the convergence will be too slow to be useful. For example, at \( x = 1 \), 199 terms only give accuracy of at most 2 significant figures. To obtain \( 10^{-6} \) accuracy we need to solve the inequality \( 4/(2n-1) < 10^{-6} \) for the first term omitted. This accuracy would require about 2 million terms!

However if we choose an \( x \) near 0 the series is useful for approximating \( \arctan x \).

Question 3 (The error function)

Solution to 3(a) The following calculations give the Taylor series for \( \text{erf}(x) \).

\[
e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}
\]

\[
e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}
\]

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^x t^{2k} dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{2k+1}}{2k+1}
\]
Solution to 3(b)  We need to evaluate erf(x) and its derivatives at x = 0. We have erf(0) = 0 and from the fundamental theorem of calculus erf′(x) = \frac{\sqrt{\pi} e^{-x^2}}{\sqrt{\pi} x}. The first few derivatives and their values at 0 are

\begin{align*}
  \text{erf}'(x) & = \frac{2}{\sqrt{\pi}} e^{-x^2} & \text{erf}'(0) & = \frac{2}{\sqrt{\pi}} \\
  \text{erf}''(x) & = -\frac{4}{\sqrt{\pi}} xe^{-x^2} & \text{erf}''(0) & = 0 \\
  \text{erf}^{(3)}(x) & = -\frac{4}{\sqrt{\pi}} (1 - 2x^2)e^{-x^2} & \text{erf}^{(3)}(0) & = -\frac{4}{\sqrt{\pi}} \\
  \text{erf}^{(4)}(x) & = -\frac{4}{\sqrt{\pi}} (-6x + 4x^3)e^{-x^2} & \text{erf}^{(4)}(0) & = 0 \\
  \text{erf}^{(5)}(x) & = -\frac{4}{\sqrt{\pi}} (-6 + 24x^2 - 8x^4)e^{-x^2} & \text{erf}^{(5)}(0) & = +\frac{24}{\sqrt{\pi}}
\end{align*}

Therefore the Taylor series expansion correct to the x^5 term is

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} x - \frac{4}{\sqrt{\pi} 3!} x^3 + \frac{24}{\sqrt{\pi} 5!} x^5 = \frac{2}{\sqrt{\pi}} \left( x - \frac{1}{3} x^3 + \frac{1}{10} x^5 \right)
\]

Solution to 3(c)  Substitute x = 1 into the Taylor polynomial

\[
T = \frac{2}{\sqrt{\pi}} \left( x - \frac{1}{3} x^3 + \frac{1}{10} x^5 - \frac{1}{42} x^7 \right)
\]

to obtain 0.8382 compared to the exact value 0.8427 correct to four decimal places obtained using MATLAB and erf(1.0). The error is $4.5 \times 10^{-3}$.

Question 4 (Bessel Functions)

Solution to 4(a)  Recall for an integral that $|\int_a^b f(x) \, dx| \leq \int_a^b |f(x)| \, dx$. Therefore

\[
|J_n(x)| \leq \frac{1}{\pi} \int_0^\pi |\cos(x \sin \theta - n\theta)| \, d\theta
\]

Since $|\cos x| \leq 1$ for all x we obtain

\[
|J_n(x)| \leq \frac{1}{\pi} \int_0^\pi d\theta = \frac{1}{\pi} = 1
\]

Solution to 4(b) and 4(c)  See the m-file bessel.m.

The following MATLAB statements compute $J_0(x), \ldots, J_{20}(x)$ using the recurrence relation $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$. 

j0 = 0.76519768655797;
j1 = 0.44005058574493;
fprintf(’%4d%25.15e %25.15e\n’,0,j0,besselj(0,1.0));
fprintf(’%4d%25.15e %25.15e
’,1,j1,besselj(1,1.0));
for n = 1 : 19
    j2 = 2*n*j1 - j0;
    j0 = j1;
    j1 = j2;
    fprintf(’%4d%25.15e %25.15e
’,n+1, j2, besselj(n+1,1) );
end

The results are

```
>> bessel
0  7.651976865579701e-001  7.651976865579666e-001
1  4.400505857449300e-001  4.400505857493366e-001
2  1.149034849319005e-001  1.149034849319005e-001
3  1.95633598266841e-002  1.95633598266841e-002
4  2.476638964109955e-003  2.476638964109955e-003
5  2.49757730212346e-004  2.49757730212346e-004
6  2.09383209284208e-005  2.09383209284208e-005
7  1.502122659313709e-006  1.502122659313709e-006
8  9.139630197108062e-008  9.139630197108062e-008
9  -3.978182777641948e-008  5.249250179911874e-009
10 -8.074692019466312e-007  2.630615123687453e-010
11 -1.610692021115620e-005  1.1980007646303138e-011
12 -3.5360379434899e-004   4.99971817948416e-013
13 -8.470381104432601e-003  1.92561676480169e-014
14 -2.198763049358041e-001  6.88540820044238e-016
15 -6.14806157098083e+000  2.29753153210353e-017
16 -1.84222108408067e+002  7.18639586807485e-019
17 -5.888959402899116e+003  2.115375568053254e-020
18 -2.000403975901619e+005  5.880344573595754e-022
19 -7.1955653842931e+006   1.548478441211653e-023
20 -2.732314430484412e+008   3.873503008524637e-025
>>
```

The exact values are given in the last column. Note that the inequality $|J_n(x)| \leq 1$ is violated by some of the approximate values.

The inequality $|J_n(x)| \leq 1$ is violated because the recurrence relation is unstable. In the calculation of $J_{n+1}(x)$ for $x = 1$ any error in $J_n(x)$ is multiplied by $2n$ so as $n$ increases the error increases rapidly.
**Question 5, Solution to Problem 1.2.13, Page 32**

\[
\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}
\]

\[
\ln(1.1) = 1.1 - \frac{(1.1)^2}{2} + \cdots + (-1)^{n-1} \frac{(1.1)^n}{n} + \cdots
\]

The error in absolute value is less than the absolute value of the first term omitted. Therefore we require

\[
\frac{(0.1)^{n+1}}{n+1} < 0.5 \times 10^{-8}
\]

which occurs for \( n = 7 \). Therefore we use the approximation

\[
\ln(1.1) \approx 1.1 - \frac{(1.1)^2}{2} + \cdots - \frac{(1.1)^7}{7}
\]

**Question 5, Solution to Computer Problem 1.2.14, Page 38**

The following MATLAB code (see m-file question5.m) implements the algorithm.

```matlab
a = 0.0;
b = 1.0;
c = 1.0/sqrt(2.0);
d = 0.25;
e = 1.0;
for k = 1 : 5
    a = b;
b = (b+c)/2;
c = sqrt(c*a);
d = d - e*((b-a)^2);
e = 2*e;
f = (b^2)/d;
g = (b+c)^2/(4*d);
fprintf('%4d%25.15e%25.15e%25.15e%25.15e
', ...
k, f, abs(f-pi), g, abs(g-pi));
end
```

The output is

<table>
<thead>
<tr>
<th>k</th>
<th>f</th>
<th>abs(f-pi)</th>
<th>g</th>
<th>abs(g-pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.187672642712108e+00</td>
<td>4.607989122315000e-02</td>
<td>3.140579250522165e+00</td>
<td>1.013403067628000e-03</td>
</tr>
<tr>
<td>2</td>
<td>3.141680293297652e+00</td>
<td>8.763970765900000e-05</td>
<td>3.141592646213352e+00</td>
<td>7.376251000000000e-09</td>
</tr>
<tr>
<td>3</td>
<td>3.141592653895444e+00</td>
<td>3.056510000000000e-10</td>
<td>3.141592653897990e+00</td>
<td>3.000000000000000e-15</td>
</tr>
<tr>
<td>4</td>
<td>3.141592653895444e+00</td>
<td>1.000000000000000e-15</td>
<td>3.141592653897990e+00</td>
<td>3.000000000000000e-15</td>
</tr>
<tr>
<td>5</td>
<td>3.141592653895444e+00</td>
<td>1.000000000000000e-15</td>
<td>3.141592653897990e+00</td>
<td>3.000000000000000e-15</td>
</tr>
</tbody>
</table>