Assignment 1, MATH/COSC 3416, Numerical Methods
Due Date: Friday, January 21, 2011
Solutions

Question 1 (the derivative experiment)

Solution See the MATLAB m-files `deriv.m`, `deriv2.m`, and `deriv3.m`. Here is the MATLAB code for question 1(b).

```
n = input('Enter number of iterations ');

exact = exp(0.5);
fprintf('Exact value = %.10e\n', exact);
fprintf('%5s %18s %18s %18s\n','n','h','r','err');
h = 1.0;
for n = 1:n
   r = (exp(0.5+h) - exp(0.5)) / h;
   err = exact - r;
   fprintf('%5d %18.10e %18.10e %18.10e\n',n,h,r,err);
   h = h / 4.0;
end
```

In all three cases the problems arise from using the limit definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

to obtain approximations to the derivative: as h approaches 0 there will be a positive but nonzero value of h such that $x+h$ is numerically equal to x. Then r will be zero.

Question 2 (the inverse tangent function)

Solution to 2(a) The following calculations give the Taylor series for $\arctan x$.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k \quad \text{which converges for } -1 < x < 1$$

$$\frac{1}{1+t^2} = \sum_{k=0}^{\infty} (-t^2)^k = \sum_{k=0}^{\infty} (-1)^k t^{2k}$$

$$\arctan(x) = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^{2k}\,dt = \sum_{k=0}^{\infty} (-1)^k \int_0^x t^{2k}\,dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{2k-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

which converges for $-1 < x \leqslant 1$.

Solution to 2(b) Since the series for $4\arctan x$ is alternating the error committed in truncating it is less than the first term omitted. Let the absolute value this term be $4x^{2n-1}/(2n-1)$. Then for $x = 1$ we require that $4/(2n-1) < 1/10^2$. This gives $n = 201$ for the first term omitted. Since the powers in the series are odd numbers then $n = 199$ is the last term so we can use the Taylor polynomial of degree 199.

Solution to 2(c) Since the series converges only on the interval $-1 < x \leqslant 1$ we expect that near the boundary of this interval the convergence will be too slow to be useful. For example, at $x = 1$, 199 terms only give accuracy of at most 2 significant figures. To obtain 10^{-6} accurracy we need to solve the inequality $4/(2n-1) < 10^{-6}$ for the first term omitted. This accuracy would require about 2 million terms!

However if we choose an x near 0 the series is useful for approximating $\arctan x$.

Question 3 (The error function)

Solution to 3(a) The following calculations give the Taylor series for $\operatorname{erf}(x)$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}\,dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^x t^{2k}\,dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{2k+1}}{2k+1}$$

Solution to 3(b) We need to evaulate $\operatorname{erf}(x)$ and its derivatives at $x = 0$. We have $\operatorname{erf}(0) = 0$ and from the fundamental theorem of calculus $\operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$. The first few derivatives and their values at 0 are

$$\begin{aligned}
\operatorname{erf}'(x) &= \tfrac{2}{\sqrt{\pi}} e^{-x^2} & \operatorname{erf}'(0) &= \tfrac{2}{\sqrt{\pi}} \\
\operatorname{erf}''(x) &= -\tfrac{4}{\sqrt{\pi}} x e^{-x^2} & \operatorname{erf}''(0) &= 0 \\
\operatorname{erf}^{(3)}(x) &= -\tfrac{4}{\sqrt{\pi}}(1-2x^2) e^{-x^2} & \operatorname{erf}^{(3)}(0) &= -\tfrac{4}{\sqrt{\pi}} \\
\operatorname{erf}^{(4)}(x) &= -\tfrac{4}{\sqrt{\pi}}(-6x+4x^3) e^{-x^2} & \operatorname{erf}^{(4)}(0) &= 0 \\
\operatorname{erf}^{(5)}(x) &= -\tfrac{4}{\sqrt{\pi}}(-6+24x^2-8x^4) e^{-x^2} & \operatorname{erf}^{(5)}(x) &= +\tfrac{24}{\sqrt{\pi}}
\end{aligned}$$

Therefore the Taylor series expansion correct to the x^5 term is

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} x - \frac{4}{\sqrt{\pi}} \frac{x^3}{3!} + \frac{24}{\sqrt{\pi}} \frac{x^5}{5!} = \frac{2}{\sqrt{\pi}} \left(x - \frac{1}{3} x^3 + \frac{1}{10} x^5 \right)$$

Solution to 3(c) Substitute $x = 1$ into the Taylor polynomial

$$T = \frac{2}{\sqrt{\pi}} \left(x - \frac{1}{3} x^3 + \frac{1}{10} x^5 - \frac{1}{42} x^7 \right)$$

to obtain 0.8382 compared to the exact value 0.8427 correct to four decimal places obtained using MATLAB and `erf(1.0)`. The error is 4.5×10^{-3}.

Question 4 (Bessel Functions)

Solution to 4(a) Recall for an integral that $|\int_a^b f(x)\,dx| \leqslant \int_a^b |f(x)|\,dx$. Therefore

$$|J_n(x)| \leqslant \frac{1}{\pi} \int_0^{\pi} |\cos(x \sin\theta - n\theta)|\, d\theta$$

Since $|\cos x| \leqslant 1$ for all x we obtain

$$|J_n(x)| \leqslant \frac{1}{\pi} \int_0^{\pi} d\theta = \frac{1}{\pi}\pi = 1$$

Solution to 4(b) and 4(c) See the m-file `bessel.m`.

The following MATLAB statements compute $J_0(x), \ldots, J_{20}(x)$ using the recurrence relation $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$.

```
j0 = 0.76519768655797;
j1 = 0.44005058574493;
fprintf('%4d%25.15e %25.15e\n',0,j0,besselj(0,1.0));
fprintf('%4d%25.15e %25.15e\n',1,j1,besselj(1,1.0));
for n = 1 : 19
   j2 = 2*n*j1 - j0;
   j0 = j1;
   j1 = j2;
   fprintf('%4d%25.15e %25.15e\n',n+1, j2, besselj(n+1,1));
end
```

The results are

```
>> bessel
   0   7.651976865579701e-001    7.651976865579666e-001
   1   4.400505857449300e-001    4.400505857449336e-001
   2   1.149034849318900e-001    1.149034849319005e-001
   3   1.956335398262982e-002    1.956335398266841e-002
   4   2.476638963888944e-003    2.476638964109955e-003
   5   2.497577284817365e-004    2.497577302112346e-004
   6   2.093832092842085e-005    2.093833800238928e-005
   7   1.502122659313709e-006    1.502325817436808e-006
   8   9.139630197108062e-008    9.422344172604498e-008
   9  -3.978182777641948e-008    5.249250179911874e-009
  10  -8.074692019466312e-007    2.630615123687453e-010
  11  -1.610960221115620e-005    1.198006746303138e-011
  12  -3.536037794434899e-004    4.999718179448416e-013
  13  -8.470381104432601e-003    1.925616764480169e-014
  14  -2.198763049358041e-001    6.885408200044238e-016
  15  -6.148066157098083e+000    2.297531532210353e-017
  16  -1.842221084080067e+002    7.186396586807485e-019
  17  -5.888959402899116e+003    2.115375568053254e-020
  18  -2.000403975901619e+005    5.880344573595754e-022
  19  -7.195565353842931e+006    1.548478441211653e-023
  20  -2.732314430484412e+008    3.873503008524637e-025
>>
```

The exact values are given in the last column. Note that the inequality $|J_n(x)| \leqslant 1$ is violated by some of the approximate values.

The inequality $|J_n(x)| \leqslant 1$ is violated because the recurrence relation is unstable. In the calculation of $J_{n+1}(x)$ for $x = 1$ any error in $J_n(x)$ is multiplied by $2n$ so as n increases the error increases rapidly.

Question 5, Solution to Problem 1.2.13, Page 32

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$$

$$\ln(1.1) = 1.1 - \frac{(1.1)^2}{2} + \cdots + (-1)^{n-1} \frac{(1.1)^n}{n} + \cdots$$

The error in absolute value is less than the absolute value of the first term omitted. Therefore we require

$$\frac{(0.1)^{n+1}}{n+1} < 0.5 \times 10^{-8}$$

which occurs for $n = 7$. Therefore we use the approximation

$$\ln(1.1) \approx 1.1 - \frac{(1.1)^2}{2} + \cdots - \frac{(1.1)^7}{7}$$

Question 5, Solution to Computer Problem 1.2.14, Page 38

The following MATLAB code (see m-file `question5.m`) implements the algorithm.

```
a = 0.0;
b = 1.0;
c = 1.0/sqrt(2.0);
d = 0.25;
e = 1.0;
for k = 1 : 5
   a = b;
   b = (b+c)/2;
   c = sqrt(c*a);
   d = d - e*((b-a)^2);
   e = 2*e;
   f = (b^2)/d;
   g = (b+c)^2/(4*d);
   fprintf('%4d%25.15e%25.15e%25.15e%25.15e\n', ...
      k, f, abs(f-pi), g, abs(g-pi));
end
```

The output is

```
1   3.187672642712108e+00   4.607998912231500e-02   3.140579250522165e+00   1.013403067628000e-03
2   3.141680293297652e+00   8.763970785900000e-05   3.141592646213542e+00   7.376251000000000e-09
3   3.141592653895444e+00   3.056510000000000e-10   3.141592653589790e+00   3.000000000000000e-15
4   3.141592653589792e+00   1.000000000000000e-15   3.141592653589790e+00   3.000000000000000e-15
5   3.141592653589792e+00   1.000000000000000e-15   3.141592653589790e+00   3.000000000000000e-15
```