Sequence spaces

Define the set of all sequences of 0’s and 1’s by
\[ \Sigma_2 = \{ s = (s_0s_1s_2 \cdots) \mid s_j \in \{0, 1\} \}. \]

Then \( \Sigma_2 \) is called the sequence space on two symbols. Sequence spaces on \( N \) symbols can be defined similarly. As examples consider

\[ s = (1011011) \quad (\text{finite}) \]
\[ t = (11011100 \cdots) \quad (s_k = 0 \text{ if } k^{\text{th}} \text{ digit of } \pi \text{ is even else } s_k = 1) \]
\[ \overline{(1)} = (1111 \cdots) \quad (\text{periodic}) \]
\[ \overline{(10)} = (101010 \cdots) \quad (\text{periodic}) \]
\[ (10111101) = (10111101101101 \cdots) \quad (\text{eventually periodic}) \]

where a bar over a finite sequence of symbols means that the sequence is repeated indefinitely. The sequence \( t \) is most likely non-terminating and not eventually periodic. We could interpret each \( s \in \Sigma_2 \) as a binary number using the correspondence

\[ (s_0s_1s_2 \cdots) \rightarrow 0.s_0s_1s_2 \cdots = \frac{s_0}{2} + \frac{s_1}{2^1} + \frac{s_2}{2^2} + \cdots = \sum_{k=0}^{\infty} \frac{s_k}{2^k} \]

which maps \( \Sigma_2 \) to the interval \([0, 1] \subset \mathbb{R} \).

Making \( \Sigma_2 \) into a metric space

In order to define the closeness of one sequence to another we need to make \( \Sigma_2 \) into a metric space.

A metric on a set \( S \) is a mapping \( d : S \times S \to \mathbb{R} \) that associates a distance \( d(s, t) \), between \( s \) and \( t \), for all \( s, t \in S \) and has the properties

1. \( d(s, t) \geq 0 \)
2. \( d(s, t) = 0 \), if and only if \( s = t \)
3. \( d(r, t) \leq d(r, s) + d(s, t) \) (triangle inequality)

A space \( S \) with a metric defined on it is called a metric space.
Example

$\mathbb{R}$ with $d(x, y) = |x - y|$ is a metric space. The proof follows from the properties of the absolute value function.

A metric for $\Sigma_2$

To make $\Sigma_2$ into a metric space define

$$d(s, t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}, \quad s, t \in \Sigma_2$$

which is suggested by the correspondence with binary numbers in $[0, 1]$.

To show that $d$ is a metric we first show that $d$ is well-defined (the sequence converges). Since $|s_i - t_i| = 0$ or $1$, we have

$$d(s, t) \leq \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

Since the geometric series converges $d$ is well-defined. Clearly $d(s, t) \geq 0$ since it is the sum of non-negative terms. Also $d(s, t) = 0$ if and only if $s_i = t_i$ for all $i$, so $d(s, t) = 0$ if and only if $s = t$. Finally, using the triangle inequality for the absolute value function

$$d(r, t) = \sum_{i=0}^{\infty} \frac{|r_i - t_i|}{2^i} \leq \sum_{i=0}^{\infty} \frac{|r_i - s_i| + |s_i - t_i|}{2^i} = d(r, s) + d(s, t)$$

The closeness theorem

We can now prove a theorem on the closeness of two sequences.

Let $s, t \in \Sigma_2$ be such that $s_i = t_i$ for $i = 0, 1, \ldots, n$. This means that $s$ and $t$ agree in their first $n + 1$ symbols. Then $d(s, t) \leq 1/2^n$. Conversely, if $d(s, t) < 1/2^n$ then $s$ and $t$ agree in their first $n + 1$ digits. In other words, the more two sequences agree in their initial symbols the closer they are.

Proof

To prove the theorem let $s_i = t_i, i \leq n$. Then

$$d(s, t) = \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n+1}} \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{2}{2^{n+1}} = \frac{1}{2^n}$$

Conversely, if $s_j \neq t_j$ for some $j \leq n$ then

$$d(s, t) \geq \frac{|s_j - t_j|}{2^j} + \sum_{i=0}^{\infty} \frac{1}{2^j} \geq \frac{1}{2^j} \geq \frac{1}{2^n}$$

But we are assuming that $d(s, t) < 1/2^n$ so we must have $s_i = t_i$ for $i \leq n$. 
**Continuous functions on** $\Sigma_2$

Let $f : \Sigma_2 \rightarrow \Sigma_2$. Then $f$ is continuous at $s \in \Sigma_2$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(f(s), f(t)) < \varepsilon$ for all $t \in \Sigma_2$ such that $d(s, t) < \delta$.

**Continuity of the shift map**

The shift map $\sigma : \Sigma_2 \rightarrow \Sigma_2$ defined by

$$\sigma(s_0s_1s_2 \cdots) = (s_1s_2s_3 \cdots)$$

is continuous everywhere.

**Proof**

To prove this let $\varepsilon > 0$ and $s = (s_0s_1s_2 \cdots)$ be given. Choose $n$ such that $1/2^n < \varepsilon$ and choose $\delta = 1/2^{n+1}$. If $d(s, t) < \delta$ then $s_i = t_i$ for $i = 0, 1, \ldots, n + 1$ by the closeness theorem. Then

$$s = (a_0a_1 \cdots a_{n+1}s_{n+2} \cdots)$$

and $t = (a_0a_1 \cdots a_{n+1}t_{n+2} \cdots)$

so

$$\sigma(s) = (a_1 \cdots a_{n+1}s_{n+2} \cdots)$$

and $\sigma(t) = (a_1 \cdots a_{n+1}t_{n+2} \cdots)$

Therefore $d(\sigma(s), \sigma(t)) \leq 1/2^n < \varepsilon$.

**Dynamics of the shift map**

Because of the simplicity of the shift map $\sigma$ it is easy to determine the periodic and eventually periodic points.

**Periodic points**

Let $s = (s_0s_1 \cdots s_{n-1}s_0s_1 \cdots s_{n-1} \cdots) = (s_0s_1 \cdots s_{n-1})$. Then for the $n$-th iterate $\sigma^n(s) = s$ so $s$ is a period-$n$ point. Since there are $2^n$ sequences $s_0s_1 \cdots s_{n-1}$ for $s_j = 0, 1$ then there are exactly $2^n$ period-$n$ points for $\sigma$.

**Eventually periodic points**

The eventually periodic points are just periodic points with some group of arbitrary leading symbols. Every eventually periodic point has the form

$$s = (e_0e_1 \cdots e_{k-1}s_0 \cdots s_{n-1})$$

in which case the iterate $\sigma^k(s)$ is periodic.
**Periodic points are dense in \( \Sigma_2 \)**

Let \( \text{Per}(\sigma) \) denote the set of all periodic points for the shift map \( \sigma \). Then \( \text{Per}(\sigma) \) is dense in \( \Sigma_2 \) if we can show for each \( s \in \Sigma_2 \) that there is a sequence \( \{\tau_n\} \) of periodic points that converges to \( s \).

To prove this let \( s = (s_0s_1s_2\cdots) \in \Sigma_2 \) be arbitrary and define \( \tau_n = (s_0\cdots s_n) \). Then \( d(\tau_n, s) \leq 1/2^n \) so we have \( \lim_{n \to \infty} \tau_n = s \).

This does not mean that “most” sequences are periodic. In fact the set of periodic points is countable but the set of non-periodic points is uncountable.

**Dense orbits**

There is a dense orbit for \( \sigma \) in \( \Sigma_3 \). This means that there is a sequence \( s^* \in \Sigma_2 \) such that some iterate of \( s^* \) is arbitrarily close to any given \( s \in \Sigma_2 \). To construct \( s^* \) let \( \omega_j \) denote the finite sequence of all blocks of symbols of length \( j \). For example,

\[
\begin{align*}
\omega_1 & = 01 \\
\omega_2 & = 0010111 \\
\omega_3 & = 00001010110101111
\end{align*}
\]

Now put this all together to construct

\[ s^* = \omega_1 \omega_2 \omega_3 \cdots \]

Then some iterate of \( s^* \) will agree with the first \( n \) symbols of any \( s \in \Sigma_2 \) for any \( n \) no matter how large.

**Summary of the shift map \( \sigma \)**

- \( \sigma \) is continuous.
- The cardinality of \( \text{Per}_n(\sigma) \) is \( 2^n \).
- \( \text{Per}_n(\sigma) \) is dense in \( \Sigma_2 \).
- The exists a dense orbit for \( \sigma \) in \( \Sigma_2 \).