SOLUTIONS

MATH 2066 EL-01 – DIFFERENTIAL EQUATIONS – TEST

Wednesday, November 3, 2010, 8:05 am – 9:20 am

Time Allowed: 75 minutes *Instructor:* Barry G. Adams

Name (PLEASE PRINT) ______________________________

Student # ______________________________

1. Answer ALL questions.
2. Total Marks: 25

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1. **(5 marks)** Find the solution $y(x)$ of the differential equation

$$\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3}$$

satisfying the initial condition $y(0) = -1/\sqrt{2}$. Simplify the solution and write it in the form $y = \phi(x)$.

Answer:

$4y^3\,dy = x(x^2+1) = x^3 + x$ integrate] 2

$y^4 = \frac{x^4}{4} + \frac{x^2}{2} + C$

$y(0) = -1/\sqrt{2} \Rightarrow C = \frac{1}{4} \Rightarrow y^4 = \frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{4}$] 2

$y^4 = \left(\frac{x^2}{2} + \frac{1}{2}\right)^2 \Rightarrow y = \pm\sqrt{\frac{x^2+1}{2}}$

Choose minus sign to satisfy initial conditions 1

$y(0) = -1/\sqrt{2}$

$y = -\sqrt{\frac{x^2+1}{2}}$

2. **(5 marks)** Solve the initial value problem

$$\frac{dy}{dt} + \frac{2}{t}y = 4t, \qquad y(1) = 2$$

for $y = \phi(t)$ and determine the interval on which the solution exists.

Answer:

$$\text{I.F.} = e^{2\int \frac{dt}{t}} = e^{2\ln t} = e^{\ln t^2} = t^2 \qquad (1)$$

$$\frac{d}{dt}\left[t^2 y\right] = 4t^3 \qquad (2)$$

$$t^2 y = t^4 + c \Rightarrow y = t^2 + \frac{c}{t^2} \qquad (1)$$

$$y(1) = 2 \Rightarrow 2 = 1 + c \Rightarrow c = 1$$

$$\boxed{y(t) = t^2 + \frac{1}{t^2} \text{ on interval } (0, \infty)} \qquad (1)$$

3. **(5 marks)** Show that the differential equation

$$(y\cos x + 2xe^y)\,dx + (\sin x + x^2e^y - 1)\,dy = 0$$

is exact and find the general solution and the solution satisfying the initial condition $y(1) = 0$.

Answer:

$M = y\cos x + 2xe^y$ $\quad N = \sin x + x^2e^y - 1$

$M_y = \cos x + 2xe^y$ $\quad N_x = \cos x + 2xe^y$ (1)

Therefore the DE is exact.

$f_x = M = y\cos x + 2xe^y$ integrate in x

$f = y\sin x + x^2e^y + h(y)$

$f_y = \sin x + x^2e^y + h'(y) = \sin x + x^2e^y - 1$ (3)

$\therefore h'(y) = -1 \Rightarrow h(y) = -y$

$\boxed{f = y\sin x + x^2e^y - y = c}$ general solution

This can also be done by starting with $f_y = N$ instead of $f_x = M$

For $y(1) = 0$ we obtain $c = 1$ so particular solution is

$\boxed{y\sin x + x^2e^y - y = 1}$ (1)

4. (a) **(2 marks)** Find the general solution of $2y'' - 5y' - 3y = 0$.
   Answer:

$2m^2 - 5m - 3 = 0 \qquad m = \frac{5 \pm \sqrt{25+24}}{4} = \frac{5}{4} \pm \frac{7}{4}$] ①

$m_1 = \frac{5}{4} - \frac{7}{4} = -\frac{1}{2}, \quad m_2 = \frac{5}{4} + \frac{7}{4} = 3$

$\boxed{y = c_1 e^{-x/2} + c_2 e^{3x}}$] ①

(b) **(2 marks)** Find the general solution in real form of $y'' - 10y' + 25y = 0$.
   Answer:

$m^2 - 10m + 25 = 0 \implies (m-5)^2 = 0$] ①

$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$] ①

(c) **(2 marks)** Find the general solution of $y'' + 4y' + 7y = 0$.
   Answer:

$m^2 + 4m + 7 = 0 \implies m = \frac{-4 \pm \sqrt{16-28}}{2} = -2 \pm \sqrt{3}\, i$] ①

$\boxed{y = e^{-2x}(c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x)}$] ①

5. **(4 marks)** Find the general solution of $y'' - y = e^x$ using the method of undetermined coefficients.

Answer:

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

$y_c = c_1 e^{-x} + c_2 e^{x}$ (1)

Let $y_p = (Ax + B)e^x$ Be^x is in y_c so omit it (1)

$y_p = Axe^x$

$y_p' = Axe^x + Ae^x = A(x+1)e^x$

$y_p'' = Ae^x + A(x+1)e^x$

$= [2A + Ax]e^x$

$y_p'' - y_p = [2A + Ax - Ax]e^x$ (2)

$= 2Ae^x = e^x \Rightarrow A = \frac{1}{2}$

particular solution is $y_p = \frac{1}{2}xe^x$

general solution is

$$\boxed{y = c_1 e^{-x} + c_2 e^x + \frac{1}{2}xe^x}$$