Differential Equations Notes, MATH 2066

Chapter Two

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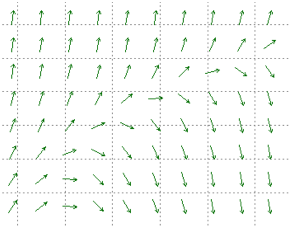
Qualitative Solutions of a First-order DE

The general first-order DE is

$$\frac{dy}{dx} = f(x, y)$$

Geometrically the function $f(x, y)$ gives the slope and direction of the solution at the point (x, y).

We can draw an arrow at each point to represent the slope and direction there



This collection of arrows is called the direction field or the slope field.

Using DFIELD to Draw Direction Fields

- Direction fields are not easily sketched by hand.
- The DFIELD applet can be used to draw a direction field for a first-order DE.
- It is available at

  `http://math.rice.edu/~dfield/dfpp.html`

- This link is also available on the course web site.
- DFIELD can also be used to plot solution curves on the direction field.
- Example: Try DFIELD with $dy/dx = 0.2xy$ which has the one parameter family of solutions $y = c\,e^{x^2/10}$.
- Example: Try DFIELD with $dy/dx = \sin y$ (exact solution ???) and find solution with initial condition $y(0) = -3/2$.

Anonymous First-order DE's

An anonymous first-order DE has the form

$$\frac{dy}{dx} = f(y)$$

The slope does not depend explicitly on the independent variable (x in this case).

Some examples are

- $dx/dt = f(x)$
- $dP/dt = kP$
- $dP/dt = P(a - bP)$
- $dA/dt = 6 - \frac{1}{100}A$
- $dT/dt = k(T - T_m)$
- $dv/dt = g - \frac{k}{m}v$

Critical Points of an Anonymous DE

- First-order anonymous DE's can be analyzed in terms of their critical points.
- For the autonomous DE $dy/dx = f(y)$ the critical points are the solutions of $f(y) = 0$.
- Critical points are also called equilibrium points or stationary points.
- At a critical point c, $dy/dx = 0$ so $y(x) = c$ is a constant solution (horizontal line in the direction field).
- For a dynamical system a first-order autonomous DE has the form $dx/dt = f(x)$ so the slope has no explicit time dependence.
- If we think of x as the position of an object at time t then at a stationary point the object is not moving ($dx/dt = 0$).

Population Example of an Autonomous DE

The autonomous DE (Verhulst or logistic model)

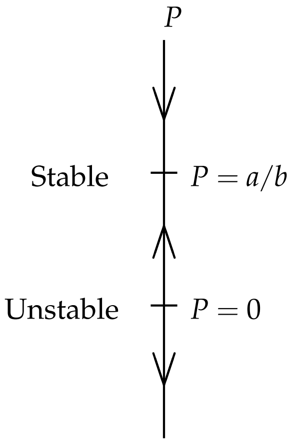
$$\frac{dP}{dt} = P(a - bP), \quad a, b > 0$$

is a population model with a limiting population.

- Here $f(P) = P(a - bP)$ so the critical points are $P = 0$ and $P = a/b$. We need to determine when $f(P) > 0$ and $f(P) < 0$.
- $f(P) > 0 \;\Rightarrow\; P > 0 \text{ and } a/b - P > 0 \;\Rightarrow\; 0 < P < a/b$
- $f(P) < 0 \;\Rightarrow\; P > 0 \text{ and } a/b - P < 0 \;\Rightarrow\; P > a/b$
- If $P < 0$ we have $f(P) < 0$. The population increases when $0 < P < a/b$ and decreases when $P > a/b$.
- This means that $P = 0$ is an unstable critical point and $P = a/b$ is a stable critical point.
- Sketch direction field, solutions, phase line and try it with DFIELD.

Phase Line

Phase line for the population DE $\frac{dP}{dt} = P(a - bP)$.



Separation of Variables

- Case 1:

$$\frac{dy}{dx} = g(x) \;\Rightarrow\; dy = g(x)dx \;\Rightarrow\; y = \int g(x)dx + c$$

- Case 2:

$$\frac{dy}{dx} = h(y) \;\Rightarrow\; \frac{dy}{h(y)} = dx \;\Rightarrow \int \frac{dy}{h(y)} = x + c$$

- Case 3:

$$\begin{aligned}\frac{dy}{dx} = g(x)h(y) &\Rightarrow\; \frac{dy}{h(y)} = g(x)dx \\ &\Rightarrow\; \int \frac{dy}{h(y)} + c_1 = \int g(x)dx + c_2\end{aligned}$$

- This last result can be expressed as $H(y) = G(x) + c$.

Example 1

Solve $(1+x)\,dy - y\,dx = 0$.

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\Rightarrow \ln|y| = \ln|1+x| + c_1$$

$$\Rightarrow e^{\ln|y|} = e^{\ln|1+x|}e^{c_1}$$

$$\Rightarrow |y| = |1+x|e^{c}$$

$$\Rightarrow y = \pm\, e^{c_1}(1+x)$$

$$\Rightarrow y = c(1+x)$$

Example 2

Solve the IVP $\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = -3$.

$$\Rightarrow y\,dy + x\,dx = 0$$

$$\Rightarrow x^2 + y^2 = c$$

This is a one parameter family of solutions.

- Substitute $x = 4, y = -3$ to get $c = 25$
- Particular solution is $x^2 + y^2 = 25$.
- Solution satisfying the initial condition is $y = -\sqrt{25 - x^2}$.
- This solution is defined on $-5 < x < 5$.

Using Maple for Examples 1 and 2

Solve $(1+x)\,dy - y\,dx = 0$.

```
> dsolve(diff(y(x),x) = y(x)/(1+x), y(x));
```

$$y(x) = \_C1\ (1+x)$$

Solve $\dfrac{dy}{dx} = -\dfrac{x}{y}, \quad y(4) = -3.$

```
> dsolve({diff(y(x),x) = -x/y(x), y(4)=-3}, y(x));
```

$$y(x) = -\sqrt{-x^2+25}$$

Note that it is necesssry to use `y(x)` not `y` in `dsolve`.

Example 3

Solve $\frac{dy}{dx} = y^2 - 4 \Rightarrow \frac{dy}{y^2-4} = dx$

Partial fractions: $\frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2}$ gives $A = 1/4$, $B = -1/4$.

$$\Rightarrow \frac{1}{4}\left[\frac{dy}{y-2} - \frac{dy}{y+2}\right] = dx$$

$$\Rightarrow \frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x + c_1$$

$$\Rightarrow \ln\left|\frac{y-2}{y+2}\right| = 4x + 4c_1 \Rightarrow \frac{y-2}{y+2} = \pm e^{4c_1}e^{4x}$$

$$\Rightarrow y - 2 = ce^{4x}(y+2) \Rightarrow y = 2\left[\frac{1+ce^{4x}}{1-ce^{4x}}\right]$$

The solutions $y = \pm 2$ are singular solutions. We can get the solution $y = 2$ by letting $c = 2$ but $y = -2$ is not part of this family.

Using Maple for Example 3

Solve $\dfrac{dy}{dx} = y^2 - 4$.

```
> dsolve(diff(y(x),x) = y(x)^2 - 4, y(x));
```

$$y\left(x\right) = -2\,\frac{1+e^{4\,x}\_C1}{e^{4\,x}\_C1-1}$$

This is eqivalent to

$$y = 2\left[\frac{1+ce^{4x}}{1-ce^{4x}}\right]$$

Example 4

Solve $(e^{2y}-y)\cos x\dfrac{dy}{dx}=e^{y}\sin 2x,\quad y(0)=0.$

$$\Rightarrow \frac{e^{2y}-y}{e^{y}}dy=\frac{\sin 2x}{\cos x}dx=2\sin x\,dx$$

$$\Rightarrow \int e^{y}dy-\int y\,e^{-y}dy=2\int \sin x\,dx$$

$$\Rightarrow e^{y}-\int y\,e^{-y}dy=-2\cos x$$

Integrate by parts ($\int u\,dv=uv-\int v\,du$) to get

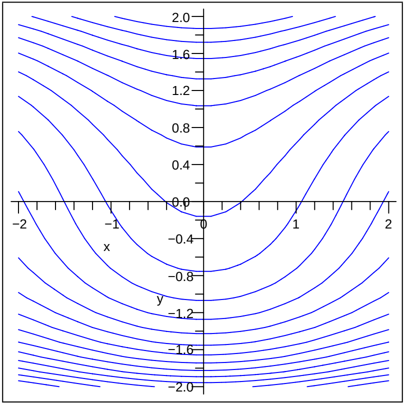
$$\int ye^{-y}\,dy=\int y\,d(-e^{-y})=-e^{-y}(1+y)$$

Solution is $e^{y}+e^{-y}(1+y)+2\cos x=c$. Substitute $y(0)=0$ to get $c=4$. Here solutions are given as level curves of the form $G(x,y)=c$. We cannot solve for y.

Contour Plot for Example 4

We cannot solve for y so we use Maple to draw a contour plot.

```
plots[contourplot](exp(y)+y*exp(-y)
   + exp(-y) + 2*cos(x), x = -2..2 ,y = -2..2,
   contours = 20, grid = [100,100]);
```



Non-uniqueness Example

Solve the IVP $\dfrac{dy}{dx} = xy^{1/2}, \quad y(0) = 0$.

$$\Rightarrow y^{-1/2}dy = x\,dx$$

$$\Rightarrow 2y^{1/2} = \frac{1}{2}x^2 + c_1$$

$$\Rightarrow y = \left(\frac{x^2}{4} + c\right)^2 \Rightarrow y = \frac{x^4}{16}, \text{ since } y(0) = 0$$

The lost solution $y = 0$ also satisfies the initial condition so we do not get uniqueness. This is expected since $\partial f/\partial y$ is not defined for $y = 0$. In fact there are infinitely many solutions of the form

$$y = \begin{cases} 0 & x < a \\ \frac{1}{16}(x^2 - a^2)^2 & x \geqslant a \end{cases} \qquad \text{one for each } a \geqslant 0.$$

Integral Definitions of Solutions

Consider the IVP $\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$.

- Recall from calculus that $\frac{dy}{dx}\int_a^x g(t)dt = g(x)$.
- Therefore the solution of the IVP can be expressed in terms of a definite integral:

$$y = \int_{x_0}^{x} g(t)dt + y_0$$

- This may be the best we can do if the integral cannot be evaluated in terms of known functions.
- In any case a numerical solution can be obtained.

The Error Function

Solve the IVP $\dfrac{dy}{dx} = e^{-x^2}, \quad y(3) = 5$.

$$\Rightarrow \int_{y(3)}^{y(x)} dy = \int_3^x e^{-t^2} dt \Rightarrow y(x) - y(3) = \int_3^x e^{-t^2} dt$$

$$\Rightarrow y(x) = 5 + \int_3^x e^{-t^2} dt$$

This integral cannot be evaluated in terms of the standard functions but it is related to the error function in probability and statistics:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Our IVP can be expressed in terms of $\mathrm{erf}(x)$ by noting that

$$\int_3^x = \int_0^x - \int_0^3 \quad \text{so} \quad \int_3^x e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \left[\mathrm{erf}(x) - \mathrm{erf}(3)\right]$$

First-order Linear DE

A first-order linear DE has the general form

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x)$$

- The left hand side is a linear combination of y' and y. If the right hand side is 0 the DE is said to be homogeneous. Otherwise it is inhomogeneous.
- Dividing by $a_1(x)$ we get the standard form

$$\frac{dy}{dx} + P(x)\,y = f(x)$$

- All solutions have the form $y = y_c + y_p$ where the complementary solution y_c is a solution of the homogeneous DE and y_p is any particular solution of the inhomogeneous DE.

Complementary and Particular Solutions

Proof that $y = y_c + y_p$.

- Let y_c satisfy the homogeneous DE $y' + P(x)\,y = 0$.
  - Then $y_c' + P(x)\,y_c = 0$.
- Let y_p satisfy the inhomogeneous DE $y' + P(x)\,y = f(x)$.
  - Then $y_p' + P(x)\,y_p = f(x)$.
- Substitute $y = y_c + y_p$ into the DE to obtain

$$\begin{aligned}&\frac{d}{dx}(y_c + y_p) + P(x)(y_c + y_p)\\ &= \underbrace{\frac{dy_c}{dx} + P(x)\,y_c}_{0} + \underbrace{\frac{dy_p}{dx} + P(x)\,y_p}_{f(x)} = f(x)\end{aligned}$$

- Therefore $y = y_c + y_p$ satisfies the inhomogeneous DE.

Solution of the Homogeneous DE

The homegeneous DE

$$\frac{dy}{dx} + P(x)\,y = 0$$

can be solved using separation of variables

$$\Rightarrow \frac{dy}{y} = -P(x)\,dx$$

$$\Rightarrow \ln|y| = -\int P(x)\,dx + c_1$$

$$\Rightarrow y = c\,e^{-\int P(x)\,dx}, \text{ where } c = \pm\,e^{c_1}$$

This solution can also be expressed as

$$\left[y\,e^{\int P(x)\,dx}\right] = c \;\Rightarrow\; \frac{d}{dx}\left[y\,e^{\int P(x)\,dx}\right] = 0$$

Solution of Inhomogeneous DE, Part 1

The inhomogeneous DE

$$\frac{dy}{dx} + P(x)\,y = f(x)$$

can be solved using the method of integrating factors. Multiply the inhomogeneous DE by $\mu(x)$ to obtain

$$\mu\frac{dy}{dx} + \mu P\,y = \mu f(x)$$

We want to choose the integrating factor μ so that $\dfrac{d}{dx}\,[\mu\, y] = \mu f(x)$.

From solution of homogeneous DE ($f(x) = 0$), we can verify that a suitable integrating factor is

$$\mu(x) = e^{\int P(x)\,dx}$$

Solution of Inhomogeneous DE, Part 2

Now integrate

$$\frac{d}{dx}\left[\mu\, y\right] = \mu f(x)$$

to obtain

$$\begin{aligned} \mu\, y &= \int \mu(x) f(x)\, dx + c \\ &\Rightarrow\; y = \frac{1}{\mu}\int \mu(x) f(x)\, dx + \frac{c}{\mu} \\ &\Rightarrow y = e^{-\int P(x)\,dx}\int e^{\int P(x)\,dx} f(x)\, dx + c\, e^{-\int P(x)\,dx} \end{aligned}$$

Another way to obtain this solution is variation of parameters.

Solution of Inhomogeneous DE, Part 3

Don't remember the complicated final result. Remember the procedure and the integrating factor.

1. Put the DE in the standard form $\dfrac{dy}{dx} + P(x)\,y = f(x)$.
2. Calculate the integrating factor $\mu(x) = e^{\int P(x)\,dx}$.
3. Multiply the DE by the integrating factor to obtain

   $$\frac{d}{dx}\left[e^{\int P(x)\,dx}\,y\right] = e^{\int P(x)\,dx} f(x)$$

   Integrate to obtain the general solution.
4. Note that the general solution y has the form $y = y_c + y_p$ where

   $$y_c = c\,e^{-\int P(x)\,dx}$$
   $$y_p = e^{-\int P(x)\,dx} \int e^{\int P(x)\,dx} f(x)\,dx$$

Example 1

Solve $\frac{dy}{dx} - 3y = 0$.

An integrating factor is $e^{-3\int dx} = e^{-3x}$

$$\Rightarrow \frac{d}{dx}\left[e^{-3x}y\right] = 0$$

$$\Rightarrow e^{-3x}y = c$$

$$\Rightarrow y = c\,e^{3x} \qquad -\infty < x < \infty$$

Example 2

Solve $2\frac{dy}{dx} - 6y = 12$.

- Put DE in standard form $\frac{dy}{dx} - 3y = 6$.
- Integrating factor is e^{-3x}.

$$\Rightarrow \frac{d}{dx}\left[e^{-3x}\, y\right] = 6e^{-3x}$$

$$\Rightarrow e^{-3x}\, y = -2e^{-3x} + c$$

$$\Rightarrow y = -2 + c\, e^{3x} \qquad -\infty < x < \infty$$

Existence and Uniqueness

Consider the general first-order DE $y' = F(x, y)$.

- Solutions exist and are unique in some region R if
  - F is continuous on R.
  - $\partial F/\partial y$ is continuous in R.
- For the linear system $y' + P(x)\,y = f(x)$ we have

$$F(x, y) = f(x) - P(x)\,y, \quad \frac{\partial F}{\partial y} = -P(x)$$

- Then we have existence and uniqueness if $f(x)$ and $P(x)$ are continuous.

Example 3

Solve $x\dfrac{dy}{dx} - 4y = x^6 e^x$ for $0 < x < \infty$.

- Standard form is $\Rightarrow \dfrac{dy}{dx} - \dfrac{4}{x}y = x^5 e^x$.
- Integrating factor is $e^{-4\int \frac{dx}{x}} = e^{-4\ln|x|} = x^{-4}$.

$$\Rightarrow \frac{d}{dx}\left[x^{-4}y\right] = x\,e^x \Rightarrow x^{-4}y = \int x\,e^x + c$$
$$\Rightarrow y = x^4 \int x\,e^x\,dx + c\,x^4$$

- Integrate by parts to get $\int x\,e^x\,dx = \int x\,d(e^x) = (x-1)e^x$.
- General solution is $y = x^4(x-1)e^x + c\,x^4$.
- $F(x,y)$ and $\partial F/\partial y$ are continuous except at $x = 0$.
- Uniqueness in any region R not containing the line $x = 0$.

Example 4

Solve $(x^2-9)\dfrac{dy}{dx}+x\,y=0$.

- Standard form is $\dfrac{dy}{dx}+\dfrac{x}{x^2-9}\,y=0$
- Integrating factor is $e^{\int\frac{x\,dx}{x^2-9}}=e^{\frac{1}{2}\ln|x^2-9|}=\sqrt{x^2-9}$

$$\begin{aligned}&\Rightarrow \frac{d}{dx}\left[\sqrt{x^2-9}\,y\right]=0\\&\Rightarrow \sqrt{x^2-9}\,y=c\\&\Rightarrow y=\frac{c}{x^2-9}\end{aligned}$$

- $x=-3$ and $x=3$ are singular points.
- Solution is unique on any of the intervals $-\infty<x<-3$, $-3<x<3$, or $3<x<\infty$.

Example 5

Solve $\frac{dy}{dx} + y = x, \quad y(0) = 4$.

- Integrating factor is e^x

$$\begin{aligned} &\Rightarrow \frac{d}{dx}\left[e^x\, y\right] = x\, e^x \\ &\Rightarrow e^x\, y = (x-1)e^x + c \ \ \text{(Integration by parts)} \\ &\Rightarrow y = x - 1 + ce^{-x} \quad \text{on } -\infty < x < \infty \end{aligned}$$

- For the initial condition $y(0) = 4$ we have $c = 5$.
- Solution is $y = x - 1 + 5e^{-x}$.
- Note that the line $y = x - 1$ is an asymptote since the term e^{-x} in the solution goes to 0 as $x \to \infty$ (transient term).

Error Function

Solve the IVP $\dfrac{dy}{dx} - 2x\,y = 2, \quad y(0) = 1.$

- The integrating factor is e^{-x^2}.

$$\Rightarrow \frac{d}{dx}\left[ye^{-x^2}\right] = 2\,e^{-x^2}$$

$$\Rightarrow y = 2\,e^{x^2}\int_0^x e^{-t^2}\,dt + c\,e^{x^2}$$

- Substitute $y(0) = 1$ to get $c = 1$.
- The error function is $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt$
- Solution of DE is

$$y = 2\,e^{x^2}\left[\frac{\sqrt{\pi}}{2}\operatorname{erf}(x)\right] + e^{x^2} = e^{x^2}\left[\sqrt{\pi}\,\operatorname{erf}(x) + 1\right]$$

Example of an Exact Differential

Consider the DE $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

- It can be expressed as $x\,dy + y\,dx = 0$.
- An expressions such as $x\,dy + y\,dx$ is called a differential.
- Since we can write $x\,dy + y\,dx = d(xy)$ we say that $x\,dy + y\,dx$ is an exact differential.
- The DE can be expressed as $d(xy) = 0$ so the solutions have the form $xy = c$ for any constant c.
- Not all differentials are exact. For example $x^2\,dx + x\,y^2\,dy$ cannot be expressed as $d(f(x,y))$ for any function $f(x,y)$.
- The general differential for functions of two variables x, y has the form

$$M(x,y)\,dx + N(x,y)\,dy$$

When is a Differential Exact?

- A differential $M(x,y)\,dx + N(x,y)\,dy$ is said to be exact if there is a function $f(x,y)$ such that

$$M(x,y)\,dx + N(x,y)\,dy = d(f(x,y))$$

- A differential $M(x,y)\,dx + N(x,y)\,dy$ is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- An exact DE has the form

$$M(x,y)\,dx + N(x,y)\,dy = d(f(x,y)) = 0$$

  so we can solve it by finding the function $f(x,y)$.
- Then the general solution of the DE is $f(x,y) = c$.

Solving an Exact DE

- An exact DE has the form

$$M(x,y)\,dx + N(x,y)\,dy = d(f(x,y)) = 0$$

- The differential of the function $z = f(x,y)$ is

$$dz = d(f(x,y)) = f_x\,dx + f_y\,dy \ \text{ where } f_x = \frac{\partial f}{\partial x} \ \text{ and } \ f_y = \frac{\partial f}{\partial y}$$

- Therefore we need to find f from the equations

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}$$

- We now show how to integrate these equations to find f in case the DE is exact.

Example 1

Solve the DE $2xy\,dx + (x^2 - 1)\,dy = 0$.

- We have $M = 2xy$ and $N = x^2 - 1$.
- The DE is exact since

$$M_y = 2x, \text{ and } N_x = 2x \Rightarrow M_y = N_x$$

- Integrate $f_x = M = 2xy$ with respect to x to get

$$\begin{aligned}
&\Rightarrow f = x^2 y + g(y) \quad \text{(integration constant depends on } y\text{)}\\
&\Rightarrow f_y = N = x^2 + g'(y) \quad \text{(differentiate with respect to } y\text{)}\\
&\Rightarrow x^2 - 1 = x^2 + g'(y)\\
&\Rightarrow g'(y) = -1 \;\Rightarrow\; g(y) = -y\\
&\Rightarrow f(x,y) = x^2 y - y
\end{aligned}$$

- Therefore the solution of the DE is $x^2 y - y = c$.

Example 1, Alternate Solution

Solve the DE $2xy\,dx + (x^2 - 1)\,dy = 0$.

- We could also get the solution by integrating with repect to y instead of x as follows.
- Integrate $f_y = N = x^2 - 1$ with respect to y to get

$$\begin{aligned}
&\Rightarrow f = x^2y - y + g(x) \quad \text{(integration constant depends on } x\text{)}\\
&\Rightarrow f_x = M = 2xy + g'(x) \quad \text{(differentiate with respect to } x\text{)}\\
&\Rightarrow 2xy = 2xy + g'(x)\\
&\Rightarrow g'(x) = 0 \;\Rightarrow\; g(x) = 0\\
&\Rightarrow f(x,y) = x^2y - y
\end{aligned}$$

- Therefore the solution of the DE is $x^2y - y = c$ as before.

Example 2

Solve the DE $(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$.

- $M_y = 2e^{2y} - \cos xy + xy\sin xy = N_x$ so the DE is exact.

$$\begin{aligned} f_y &= 2xe^{2y} - x\cos xy + 2y \quad \text{(Integrate with respect to } y\text{)} \\ f &= 2x\int e^{2y}\,dy - x\int \cos xy\,dy + 2\int y\,dy \\ &= xe^{2y} - \sin xy + y^2 + h(x) \quad \text{(int constant depends on } x\text{)} \end{aligned}$$

- Differentiate with respect to x to get

$$f_x = e^{2y} - y\cos xy + h'(x)$$

- But $f_x = M = e^{2y} - y\cos xy$ so choose $h'(x) = 0$ and $h(x) = 0$.
- Family of solutions is $x^{2y} - \sin xy + y^2 = c$.

Integrating Factors

- Sometimes a non-exact DE can be made exact by muliplying it by an integrating factor.
- Let $\mu(x, y)$ be an integrating factor and multiply the DE $M\,dx + N\,dy = 0$ by μ to obtain

  $$\mu M\,dx + \mu N\,dy = 0$$

  To be exact we must have $(\mu M)_y = (\mu N)_x$.
- There are two useful cases:
  - μ and $\frac{M_y - N_x}{N}$ are functions of x only. Then

    $$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{M_y - N_x}{N} \text{ gives an integrating factor}$$

  - μ and $\frac{N_x - M_y}{M}$ are functions of y only. Then

    $$\frac{1}{\mu}\frac{d\mu}{dy} = \frac{N_x - M_y}{M} \text{ gives an integrating factor}$$

Integrating Factor Example

Solve the DE $xy\,dx + (2x^2 + 3y^2 - 20)\,dy$.

- We have $M_y = x$ and $N_x = 4x$ so the DE is not exact

$$\frac{M_y - N_x}{N} = \frac{-3x}{2x^2 + 3y^2 - 20} \quad \text{is not a function of } x$$

$$\frac{N_x - M_y}{M} = \frac{3x}{xy} = \frac{3}{y} \quad \text{is a function of } y$$

- Then integrating factor satisfies

$$\frac{1}{\mu}\frac{d\mu}{dy} = \frac{3}{y} \;\Rightarrow\; \ln|\mu| = 3\ln|y| \Rightarrow \mu = y^3$$

- The exact DE is

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0$$

Solution by Substitution

There are many special methods for solving first order DE's. We consider only two here.

- Homogeneous DE's
- Bernoulli's Equation

Bernoulli Equation

The Bernoulli equation is $\frac{dy}{dx} + P(x)y = f(x)y^n$.

- It is linear if $n = 1$ and can be converted to a linear DE for w as a function of x with the substitution $y = w^k$ for some k.
- Make the substitution $y = w^k$ and $y' = kw^{k-1}w'$.

$$\begin{aligned} &\Rightarrow kw^{k-1}w' + Pw^k = fw^{kn} \quad \text{(now divide by } w^{k-1}\text{)} \\ &\Rightarrow kw' + Pw = fw^{k\,n-k+1} \\ &\Rightarrow kw' + Pw = f \quad \text{(if } k\,n - k + 1 = 0\text{)} \end{aligned}$$

- Therefore choose $k = \frac{1}{1-n}$ to get the linear DE

$$k\frac{dw}{dx} + P(x)w = f(x)$$

- Solve this for w and then let $y = w^k$.

Example

Solve $\frac{dy}{dx} + y = xy^3$.

- Let $y = w^k$ to get

$$\begin{aligned} & kw^{k-1}w' + w^k = xw^{3k} \quad \text{(divide by } w^{k-1}\text{)} \\ & \Rightarrow kw' + w = xw^{2k+1} \quad \text{(choose } k = -1/2\text{)} \\ & \Rightarrow w' - 2w = -2x \end{aligned}$$

- Now you can solve this DE and substitute $w = y^{-2}$.
- This returns to the variables x and y.