Differential Equations Notes, MATH 2066

Chapter One

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Classification of Differential Equations

- A differential equation (DE) is an equation containing one or more derivatives of an unknown function.
    - Ordinary DE (ODE), partial DE (PDE)
    - linear DE or non-linear DE
    - 1st order DE, 2nd order DE, ...
- The goal is to find the unknown function
- We consider only ordinary DE's in this course.
- Systems of DE's (2 or more DE's)

Examples of an Ordinary Differential Equation

- General first order DE

$$\frac{dy}{dx} = f(x, y)$$

- A linear homogeneous first-order ODE

$$\frac{dx}{dt} + P(t)x = 0$$

- A linear inhomogeneous second-order ODE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = f(x)$$

- A second-order non-linear ODE (ω is a parameter)

$$\frac{d^2x}{dt^2} + \omega^2 x^2 = f(t)$$

Example of a System of two ODE's

$$\begin{aligned} \frac{dx}{dt} &= f(t,x,y) \\ \frac{dy}{dt} &= g(t,x,y) \end{aligned}$$

- Here there are two dependent variables, x and y.
- Each DE involves both of these variables.
- A solution has the form $x = \phi(t)$ and $y = \psi(t)$ on some interval $a < t < b$.

Solution of a DE

- Our goal is to find one or more solutions of a DE
- If a DE has the form (general n-th order DE)

  $$F(x, y, y', \ldots, y^{(n)}) = 0$$

  then $y = \phi(x)$ is a solution if

  $$F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0$$

  on some interval I.
- Example: $y = e^{x^2/10}$ is a solution of $\dfrac{dy}{dx} = \dfrac{1}{5}xy$ on $(-\infty, \infty)$.
- Verify:

  $$\begin{aligned} \text{LHS of DE} &= \frac{dy}{dx} = \frac{2x}{10}e^{x^2/10} = \frac{x}{5}e^{x^2/10} \\ \text{RHS of DE} &= \frac{1}{5}xy = \frac{x}{5}e^{x^2/10} = \text{LHS} \end{aligned}$$

Graph of a Solution Curve

- The graph of a solution $y = \phi(x)$ is called a solution curve but note that a function and a solution may not be the same since they can have different domains.
- Example: $y = 1/x$ is a solution of $xy' + y = 0$ on any interval of the x-axis not containing 0 (verify this).
- As a function $y = 1/x$ is defined everywhere on the x-axis except at $x = 0$ (hyperbola).
- A solution is always defined on an interval
- We could define the domain of the solution on either branch of the hyperbola, $(-\infty, 0)$ or $(0, \infty)$.

Explicit and Implicit Solutions

- Explicit solution example
  - $y = 1/x$ is an explicit solution of $xy' + y = 0$ since it has the explicit form $y = \phi(x)$.
- Implicit solution example
  - $x^2 + y^2 = 25$ is an implicit solution of $\frac{dy}{dx} = -\frac{x}{y}$. To verify this use implicit differentiation of $x^2 + y^2 = 25$ to get

$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}$$

  - This implicit solution represents a circle which defines two explicit solutions on the interval $-5 < x < 5$:
  - The upper half of the circle $y = \sqrt{25 - x^2}$.
  - The lower half of the circle $y = -\sqrt{25 - x^2}$.

Families of Solutions

- If a solution contains one or more parameters we say that it represents a family of solutions, one solution for each value of the parameters.
- Solutions not belonging to a family are said to be singular.
- Example: For any $c > 0$, $x^2 + y^2 = c$ gives a one parameter family of solutions of $\dfrac{dy}{dx} = -\dfrac{x}{y}$.
- Example: The DE $\dfrac{dy}{dx} = xy^{1/2}$ has the family of solutions

  $$y = \left(\frac{1}{4}x^2 + c\right)^2 \quad \text{on} \quad (-\infty, \infty)$$

  for any value of the parameter c.
- The solution $y = 0$ for all x is not obtained for any value of c so it is a singular solution.

Initial Value Problems

- An initial value problem (IVP) for a first order DE is the DE and an initial condition:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

- An IVP for a second-order DE is the DE and two initial conditions

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y_{(}x_0) = y_0, \ y'(x_0) = y_1$$

- For a dynamical system the second-order IVP would be

$$\frac{d^2x}{dt^2} = f(t, y, y'), \quad y_{(}t_0) = y_0, \ y'(t_0) = y_1$$

- Here we imagine a particle moving in time whose initial position and velocity are given at time t_0.

IVP Example 1

- $y' = y$ has family of solutions

$$y = ce^x$$

- Find the solution satisfying $y(0) = 3$.

$$y(0) = 3$$

$$\Rightarrow\ 3 = ce^0 \ \Rightarrow\ c = 3 \ \Rightarrow\ y = 3e^x$$

- Find the solution satisfying $y(1) = -2$.

$$y(1) = -2$$

$$\Rightarrow\ c = -2e^{-1} \ \Rightarrow\ y = -2e^{-1}e^x \ \Rightarrow\ y = -2e^{x-1}$$

IVP Example 2

- $y' + 2xy^2 = 0$ has family of solutions

$$y = \frac{1}{x^2 + c}$$

- Find the solution satisfying $y(0) = -1$.

$$y(0) = -1 \;\Rightarrow\; -1 = c \;\Rightarrow\; y = \frac{1}{x^2 - 1}$$

- As a function, y is defined everywhere except at $x = \pm 1$
- As a solution of the DE, y could be defined on any of the intervals $(-\infty, -1)$, $(-1, 1)$, or $(1, \infty)$.
- As a solution of the IVP, y is defined only on the interval $(-1, 1)$ since the initial value 0 of x is in this interval.

IVP Example 3

- The second order DE $x'' + 16x = 0$ has a 2 parameter family of solutions

$$x = c_1 \cos 4t + c_2 \sin 4t$$

- Find the solution for the IVP $x(\pi/2) = -2$, $x'(\pi/2) = 1$.

$$\begin{aligned} x(\pi/2) = -2 &\Rightarrow c_1 \cos 2\pi + c_2 \sin 2\pi = -2 \Rightarrow c_1 = -2 \\ x'(\pi/2) = 1 &\Rightarrow -4c_1 \sin 2\pi + 4c_2 \cos 2\pi = 1 \Rightarrow c_2 = 1/4 \end{aligned}$$

- The solution of the IVP, defined on $(-\infty, \infty)$, is

$$x = -2 \cos 4t + \frac{1}{4} \sin 4t$$

Existence and Uniqueness of Solutions

In applications of DE's to modeling it is essential that solutions of IVP's exist and are unique.

- For the first order IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

- Let R be a rectangular region $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ containing the point (x_0, y_0)
- Let $f(x, y)$ and $\dfrac{\partial f(x, y)}{\partial y}$ be continuous on R.
- Then there exists an interval $I_0 = (x_0 - h, x_0 + h)$, $h > 0$, contained in (a, b) and a unique function $y = \phi(x)$ defined on I_0 that is the solution of the initial value problem.

Remarks

- The theorem gives sufficient but not necessary conditions for uniqueness so it is possible to have IVP’s that are unique even though the conditions of the theorem are not satisfied.
- Thus the theorem cannot be used to prove non-uniqueness.

Uniqueness Examples

- $\frac{dy}{dx} = y$
  - Here $f(x, y) = y$ and $\frac{\partial f}{\partial y} = 1$
  - These are continuous everywhere so solution is unique for any initial condition $y(x_0) = y_0$.
- $\frac{dP}{dt} = P(a - bP)$
  - Here $f(t, P) = P(a - bP)$ and $\frac{\partial f}{\partial P} = a - 2bP$.
  - These are continuous everywhere so solution is unique for any initial condition $P(t_0) = P_0$.

Non-uniqueness Example

- $\frac{dy}{dx} = xy^{1/2}$
- Here $f(x, y) = xy^{1/2}$ and $\frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$ which is not continuous at $y = 0$, the x-axis.
- For the initial condition $y(2) = 1$ the solution is unique
- For the initial condition $y(0) = 0$ the conditions of theorem are not satisfied so theorem says nothing about uniqueness.
- In fact the solutions $y = 0$ and $y = x^4/16$ both satisfy $y(0) = 0$ so there is no unique solution satisfying $y(0) = 0$.

Simple Population Dynamics

- Simplest model assumes that the change in population per unit time is proportional to the population at time t.

$$\frac{dP}{dt} = kP(t), \quad k > 0, \quad P(0) = P_0$$

- This model is valid for short periods of time but the long term behaviour is a population that increases forever.
- Does not account for limiting factors such as limited resources

Simple Radioactive Decay

- Simplest model assumes that the decrease in the amount of a radiactive substance per unit time is proportional to the amount present at time t.

$$\frac{dA}{dt} = kA(t), \quad k < 0, \quad A(0) = A_0$$

- This model is the same as the previous population model except that now the constant k is negative since the amount is decreasing

Newton's Law of Cooling/Warming

- Let T_m be the constant temperature of the surrounding medium (ambient temperature).
- Let T be the temperature of an object (T_0 at time 0)
- Then a model for the change in temperature T with time is

$$\frac{dT}{dt} = k(T - T_m), \quad k < 0, \quad T(0) = T_0$$

If $T > T_m$ then $dT/dt < 0$, corresponding to cooling
If $T < T_m$ then $dT/dt > 0$, corresponding to heating

- The medium should be large compared to the object so that T_m is a constant.

Simple Model for the Spread of a Disease

- $x(t)$ is number of people having the disease at time t.
- $y(t)$ is number of people not yet exposed at time t.
- xy represents the interaction between infected and uninfected people.
- Rate at which the disease spreads is

$$\frac{dx}{dt} = kxy$$

- If we start with n uninfected people and introduce 1 infected person then $x + y = n + 1$ so the IVP is

$$\frac{dx}{dt} = kx(n + 1 - x), \quad k > 0, \quad x(0) = 1$$

Chemical Reaction Example

Chemical reaction is $A + B \longrightarrow C$

- X is the amount of C formed at time t.
- α is the initial amount of A at time 0
- β is the initial amount of B at time 0
- $\alpha - X$ is amount of A not yet converted to C
- $\beta - X$ is amount of B not yet converted to C
- The IVP is

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X), \quad k > 0, \quad X(0) = 0$$

Mixtures in a Tank

- Tank contains 300 gal of brine containng 10 lb of salt initially.
- Brine containing 2 lb/gal of salt flows in at 3 gal/min.
- Brine at 3 gal/min flows out.
- $A(t)$ is the number of lb of salt in the tank at time t.
- The rate of change of A in lb per minute is given by
- $\dfrac{dA}{dt} = \text{input salt rate} - \text{output salt rate} = R_{in} - R_{out}$
- $R_{in} = \frac{2\,lb}{gal} \cdot \frac{3\,gal}{min} = \frac{6\,lb}{min}, \quad R_{out} = \frac{A(t)\,lb}{300\,gal} \cdot \frac{3\,gal}{min} = \frac{A(t)\,lb}{100\,min}$
- The IVP is

$$\frac{dA}{dt} + \frac{A}{100} = 6, \quad A(0) = 10$$

Draining a Tank

Here we have a tank of water that is being drained by a hole in the bottom of the tank.

- A_w is the constant cross sectional area of the tank.
- A_h is the cross sectional area of the hole.
- h is the height of water in the tank at time t.
- V is the volume of water in the tank at time t.
- From Torricelli's law the speed of exit through the hole is $v = \sqrt{2gh}$.
- Therefore $dV/dt = -A_h \sqrt{2gh}$
- But $V = A_w h$ so $dV/dt = A_w dh/dt = -A_h \sqrt{2gh}$
- Therefore the DE for the height of the water at time t is

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}, \quad h(0) = h_0$$

Simple Electric Circuit

Consider a series circuit containing a resistor, a capacitor, an inductor and a voltage source. The current is $i = dq/dt$ (amperes) where q is the charge (coulombs).

- inductance L (henries, h) gives voltage drop $L\frac{di}{dt} = L\frac{d^2q}{dt^2}$.
- resistance R (ohms, Ω) gives voltage drop $iR = R\frac{dq}{dt}$.
- capacitance C (farads, f) gives voltage drop q/C.
- The DE is obtained by setting the sum of the voltage drops to $E(t)$ the voltage source.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

Mass on a Spring with Damping

The DE for a mass m on a spring with damping and an external force is

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

where x is the displacement, $2\lambda = \beta/m$, $\omega^2 = k/m$, β is the damping coefficent and k is the spring constant.

This DE has the same form as the electric circuit DE

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

Falling Object Without Air Resistance

From Newton's second law force is mass times acceleration

$$F = ma = m\frac{dv}{dt} \;\Rightarrow\; m\frac{d^2x}{dt^2} = mg$$

so the first-order IVP for velocity is

$$\frac{dv}{dt} = g, \quad v(0) = v_0$$

and the second-order IVP for position is

$$\frac{d^2x}{dt^2} = g, \quad x(0) = x_0, \quad x'(0) = v_0$$

Here we assume that the positive x direction is downward.

Falling Object With Air Resistance

Now the first-order IVP for velocity is

$$m\frac{dv}{dt} = mg - kv, \quad v(0) = v_0$$

- k is a constant and we are assuming that air resistance is proportional to the velocity.
- We assume that the positive x direction is downward.
- There are other posibilities. For example we could assume that air resistance is proportional to v^2 or some other power of v.