
Notes for MATH/COSC 1056

Mathematical Induction

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B.G.Adams

An important proof technique is **mathematical induction**. A proof by induction begins with a propositional function $S(n)$ defined for all integers $n \geq m$ for some smallest integer m . The goal is to show that $S(n)$ is true for all integers $n \geq m$.

First we try to show that the statement $S(n)$ is true for the first case $n = m$. Then we use mathematical induction to show that $S(n)$ is true for all $n \geq m$. In numerical examples it is usually necessary to guess a formula for the first few values of n and then use induction to verify that the formula is correct for all values of n .

Standard Example (sum of the first n integers)

Consider the problem of calculating the sum of the first n integers. Define $S_n = 1 + 2 + 3 + \dots + n$ for $n \geq 1$. Then we have the following results for the first few cases.

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + 2 = 3 \\ S_3 &= 1 + 2 + 3 = 6 \\ S_4 &= 1 + 2 + 3 + 4 = 10 \end{aligned}$$

At this stage we might be clever enough to notice that the values of the sums are $n(n+1)/2$ for $n = 1, 2, 3, 4$. For example, $4(4+1)/2 = 10$. Later we will show how to justify this guess.

We would like to show that this formula is true for all values of $n \geq 1$. This means we must show that the propositional function S defined by the statements

$$S(n) : S_n = \frac{n(n+1)}{2}$$

is true for all $n \geq 1$. But how do we prove an infinite number of statements? We can do this using mathematical induction.

The first statement $S(1)$ is true since $1 = 1(1+1)/2$. Suppose we can show for an arbitrary n that $S(n) \rightarrow S(n+1)$. Then we can reason as follows

$S(1)$ is true,
 $S(2)$ is true since $S(1) \rightarrow S(2)$ and $S(1)$ is true (modus ponens),
 $S(3)$ is true since $S(2) \rightarrow S(3)$ and $S(2)$ is true (modus ponens),
 $S(4)$ is true since $S(3) \rightarrow S(4)$ and $S(3)$ is true (modus ponens),
 ...

Thus, since $S(1)$ is true, we can prove an infinite number of statements if we can show that for an arbitrary $n > 1$

$$S(n+1) : S_{n+1} = \frac{(n+1)(n+2)}{2} \text{ follows from } S(n) : S_n = \frac{n(n+1)}{2}$$

Here the formula for S_{n+1} is obtained by substituting $n + 1$ for n in the formula for S_n . This formula is easily proved as follows

$$\begin{aligned} S_{n+1} &= 1 + 2 + \cdots + n + 1 = (1 + 2 + \cdots + n) + n + 1, \\ &= \frac{n(n+1)}{2} + n + 1, \text{ since we are assuming } 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \\ &= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}. \end{aligned}$$

This shows that $S(n) \rightarrow S(n+1)$.

This process is called mathematical induction and is a two step process for proving an infinite number of statements: (1) prove the base case, (2) prove the inductive case ($S(n) \rightarrow S(n+1)$).

Where does the formula for S_n come from?

If you have trouble guessing the formula for the sum of the first n integers here is a trick: Write the sum and beneath it write the sum backwards, then add the results:

$$\begin{array}{r} S_n = 1 + 2 + 3 + \cdots + n \\ S_n = n + n-1 + n-2 + \cdots + 1 \\ \hline 2S_n = n+1 + n+1 + n+1 + \cdots + n+1 \end{array}$$

Here $n + 1$ occurs n times so $2S_n = n(n+1)$ and $S_n = n(n+1)/2$.

Principle of Mathematical Induction (Weak form)

Given an integer m , suppose that $S(n)$ is a statement for each integer $n \geq m$ and suppose that

1. $S(m)$ is true,
(BASIS STEP OR CASE)
2. if $S(n)$ is true for an arbitrary integer $n \geq m$ then $S(n+1)$ is true.
(INDUCTIVE STEP OR CASE)

Then $S(n)$ is true for every $n \geq m$.

We can express this as the following valid argument with an infinite number of hypotheses:

$$\begin{array}{l} S(m) \\ \forall n \geq m, S(n) \rightarrow S(n+1) \\ \hline \therefore \forall n \geq m, S(n) \end{array}$$

Recursive and non-recursive definitions of $n!$

We will need the following definitions of $n!$. The non-recursive definition is

$$n! = \begin{cases} 1, & \text{if } n = 0 \\ 1 \cdot 2 \cdots (n-1) \cdot n, & \text{if } n > 0 \end{cases}$$

For example, $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$. The recursive definition is

$$n! = \begin{cases} 1, & \text{if } n = 0 \\ n(n-1)!, & \text{if } n > 0 \end{cases}$$

Inequality example

Many applications of induction involve inequalities. For example, we can use induction to prove that $n! \geq 2^{n-1}$ for $n \geq 1$.

Statements to prove $S(n) : n! \geq 2^{n-1}, n \geq 1$.

Basis step $S(1)$ is true since $1! = 1, 2^0 = 1$ so $1! \geq 1$.

Inductive step Let us assume as the inductive step that for arbitrary $n > 1$, $S(n) : n! \geq 2^{n-1}$ is true. We need to show that the statement $S(n+1) : (n+1)! \geq 2^n$, obtained by substituting $n+1$ for n in $S(n)$, is true. This can be done as follows using the inductive hypothesis and the recursive definition of $n!$:

$$\begin{aligned} (n+1)! &= (n+1)n!, \text{ recursive definition} \\ &\geq (n+1)2^{n-1}, \text{ since we are assuming } n! \geq 2^{n-1} \\ &\geq 2 \cdot 2^{n-1}, \text{ since } n+1 \geq 2 \text{ for } n \geq 1 \\ &= 2^n, \text{ using law of exponents } x^a x^b = x^{a+b} \end{aligned}$$

Therefore $S(n) \rightarrow S(n+1)$ so $S(n)$ is true for all $n \geq 1$. [We have used a basic property of inequalities here: if $a \geq b$ and $c \geq 0$ then $ac \geq bc$. In our case $a = (n+1)$, $b = 2$ and $c = 2^{n-1}$.]

Geometric series example

The geometric series with initial term a and term ratio r is defined by $S_n = a + ar + ar^2 + \cdots + ar^n$. We can derive a formula for the sum as follows

$$\begin{array}{rcl} S_n & = & a + ar + ar^2 + \cdots + ar^n \\ rS_n & = & ar + ar^2 + \cdots + ar^n + ar^{n+1} \\ \hline (1-r)S_n & = & a - ar^{n+1} \end{array}$$

where we have subtracted the second equation from the first. All but the first and last terms in the sums cancel. Therefore for $r \neq 1$ we have the sum formula

$$S_n = \frac{a(1-r^{n+1})}{1-r} = \frac{a(r^{n+1}-1)}{r-1}$$

We can prove this formula by induction (assuming $r \neq 1$):

Statements to prove $S(n) : S_n = a(r^{n+1} - 1)/(r - 1)$ for $n \geq 0$.

Basis step For $n = 0$ we have $S_0 = a$ and $a(r - 1)/(r - 1) = a$ so $S(0)$ is true.

Inductive step Assume for arbitrary $n > 0$ that $S(n)$ is true. Then we can show that $S(n + 1) : S_{n+1} = a(r^{n+2} - 1)/(r - 1)$ is true as follows:

$$\begin{aligned}
 S_{n+1} &= a + ar + ar^2 + \cdots + ar^n + ar^{n+1} \\
 &= (a + ar + ar^2 + \cdots + ar^n) + ar^{n+1} \\
 &= S_n + ar^{n+1} \\
 &= \frac{a(r^{n+1} - 1)}{r - 1} + ar^{n+1}, \text{ (using induction hypothesis)} \\
 &= \frac{ar^{n+1} - a + ar^{n+2} - ar^{n+1}}{r - 1}, \text{ (after taking common denominator)} \\
 &= \frac{a(r^{n+2} - 1)}{r - 1}
 \end{aligned}$$

Therefore $S(n) \rightarrow S(n + 1)$ so $S(n)$ is true for all $n \geq 1$. For $r = 1$ the sum is $S_n = (n + 1)a$ [Prove it using induction].

A divisibility example

Consider the numbers $5^n - 1$ for $n \geq 1$. For $n = 1, 2, 3$ we get the numbers 4, 24, 124. Continuing we notice that these numbers are all divisible by 4. This suggests that we try to prove that $5^n - 1$ is divisible by 4 for all $n \geq 1$. To say that a number m is divisible by 4 means that $m = 4k$ for some integer k so we want to prove that for each $n \geq 1$ there is an integer k such that $5^n - 1 = 4k$.

Statements to prove $S(n)$: For each $n \geq 1$, $5^n - 1 = 4k$ for some integer k .

Basis step For $n = 1$ we have $5^1 - 1 = 4 = 4k$ for $k = 1$ so $S(1)$ is true.

Inductive step Assume for arbitrary $n > 1$ that $S(n)$ is true. Then we can show that $S(n + 1) : 5^{n+1} - 1 = 4m$ is true for some integer m as follows:

$$\begin{aligned}
 5^{n+1} - 1 &= 5 \cdot 5^n - 1 \text{ (from induction hypothesis } 5^n - 1 = 4k, \text{ or } 5^n = 4k + 1) \\
 &= 5(4k + 1) - 1 \\
 &= 20k + 4 \\
 &= 4(5k + 1) \\
 &= 4m \text{ where } m = 5k + 1
 \end{aligned}$$

Therefore $S(n) \rightarrow S(n + 1)$ so $S(n)$ is true for all $n \geq 0$.

Principle of Mathematical Induction (Strong form)

There is a stronger form of mathematical induction which can be shown to be equivalent to the weak form:

Given an integer m , suppose that $S(n)$ is a statement for each integer $n \geq m$ and suppose that

1. $S(m)$ is true,
(BASIS STEP OR CASE)
2. if $S(m), S(m+1), \dots, S(n-1)$ are true then $S(n)$ is true,
(INDUCTIVE STEP OR CASE)

Then $S(n)$ is true for every $n \geq m$.

We can express this as the following valid argument with an infinite number of hypotheses:

$$\begin{array}{l}
 S(m) \\
 \forall n > m, S(m), S(m+1), \dots, S(n-1) \rightarrow S(n) \\
 \hline
 \therefore \forall n \geq m, S(n)
 \end{array}$$

A postage stamp example

For $n \geq 4$ consider the statement “ $S(n)$: Postage can be made for n cents using only 2 cent and 5 cent stamps”. Try a few cases

$$\begin{array}{l}
 S(4) : 4 = 2 + 2 \\
 S(5) : 5 = 5 \\
 S(6) : 6 = 2 + 2 + 2 \\
 S(7) : 7 = 5 + 2 \\
 S(8) : 8 = 2 + 2 + 2 + 2 \\
 S(9) : 9 = 5 + 2 + 2
 \end{array}$$

We can prove the general case using the strong form of induction.

basis step For $n = 4$ we can make postage using two 2 cent stamps and for $n = 5$ we can make postage using one 5 cent stamp.

inductive step Now assume for arbitrary $n \geq 6$ that postage for k cents can be made using only 2 cent and 5 cent stamps for $6 \leq k < n$ (so we assume $S(6), S(7), \dots, S(n-1)$ are true).

To make postage for n cents we can first make postage for $n-2$ cents (induction assumption) and then add a 2 cent stamp to make postage for n cents.

Homework

1. Find formulas for the sums

$$E_n = 2 + 4 + 6 + \cdots + 2n \quad (\text{sum of even integers})$$

$$O_n = 1 + 3 + 5 + \cdots + 2n - 1 \quad (\text{sum of odd integers})$$

and prove them using induction.

2. Prove that

$$\sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. Prove that

$$\sum_{k=1}^n k(k+1) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

4. Prove that

$$\sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

5. Prove that

$$\sum_{k=1}^n \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

6. Prove that $2n + 1 \leq 2^n$, for $n \geq 3$.

7. Prove that $1 + nx \leq (1 + x)^n$, $\forall x > -1, \forall n \geq 2$.

8. Prove that $7^n - 1$ is divisible by 6, $\forall n \geq 1$.

9. Prove that $n^3 - 7n + 3$ is divisible by 3, $\forall n \geq 2$.

10. Prove that $3^{2n} - 1$ is divisible by 8, $\forall n \geq 1$.